Solutions to Physics 108 Final (Spring of 2018)

1-(1) For a real object $s_o > 0$; for a real image, $s_i > 0$. To have a linear magnification of 10, we need

$$|m| = \left| -\frac{s_i}{s_o} \right| = 10$$

$$\therefore \quad s_i = 10\, s_o$$

From

$$\frac{1}{s_o} + \frac{1}{s_i} = \frac{1}{f}$$

$$\Rightarrow \frac{1}{s_o}\left(1 + \frac{1}{10}\right) = \frac{1}{f}$$

$$\therefore s_o = \frac{11}{10} f = 11\,\text{cm}$$ ※ (Object in front).

1-(2) When using $f = +10$ cm as a magnifying glass the achievable angular magnification is to have $s_i = -d_o = -25$ cm. Then

$$m_A = 1 + \frac{d_o}{f} = 1 + \frac{25\,\text{cm}}{10\,\text{cm}} = 3.5$$ ※

1-(3) For a real image, $s_i > 0$; for a virtual object, $s_o < 0$. To achieve a magnification of 10, we need

$$|m| = \left|-\frac{s_i}{s_o}\right| = 10$$

Thus

$$s_i = -10 s_o$$

From

$$\frac{1}{s_o} + \frac{1}{s_i} = \frac{1}{f}$$

we have

$$\frac{1}{s_o}\left(1 - \frac{1}{10}\right) = -\frac{1}{10}$$

$$\therefore s_o = -9 \text{ cm} = \frac{9}{10} f \quad \text{(Object behind)}$$

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2-(1) Angular magnification for a telescope is

$$M_A = \frac{f_o}{f_e} = \frac{150\,cm}{2.5\,cm} = 60$$

2-(2) The smallest angle that it can resolve due to diffraction is

$$\delta\alpha = \frac{1.22\lambda}{D} = 1.22 \times \frac{0.5\times10^{-6}\,m}{9\times10^{-2}\,m} = 6.8\times10^{-6}\,rad.$$

2-(3) The smallest separation is given by

$$\delta x = L\cdot\delta\alpha = 6.8\times10^{-6}\,rad \times 4\times10^{8}\,m$$

$$= 2.7\times10^{3}\,m$$

3-(1) For a single-slit,

$$I_1(y) = I_1(\theta) = I_{inc} \frac{d^2}{\sqrt{\lambda R_0}} \frac{\sin^2\left(\frac{\pi d}{\lambda}\sin\theta\right)}{\left(\frac{\pi d}{\lambda}\sin\theta\right)}$$

$$R_0 = (L^2 + y^2)^{1/2}$$

$$\sin\theta = \frac{y}{\sqrt{L^2+y^2}} = \frac{y}{R_0}$$

3-(2) The intensities from all three slits are the same individually, i.e.,

$$I_1(y) = I_2(y) = I_3(y) \equiv I_1(\theta)$$

But when all three slits are open, there are additionally three interference terms:

$$\begin{aligned} I_{total}(\theta) &= I_1(\theta) + I_2(\theta) + I_3(\theta) \\ &+ 2\sqrt{I_1(\theta) I_2(\theta)} \cos(\phi_2 - \phi_1) \\ &+ 2\sqrt{I_1(\theta) I_3(\theta)} \cos(\phi_3 - \phi_1) \\ &+ 2\sqrt{I_2(\theta) I_3(\theta)} \cos(\phi_2 - \phi_1) \end{aligned}$$

$$= 3I_1(\theta) + 2I_1(\theta)\Big[\cos(\phi_2-\phi_1) + \cos(\phi_3-\phi_1) + \cos(\phi_3-\phi_2)\Big]$$

where

$$\phi_2-\phi_1 = \frac{2\pi a}{\lambda}\sin\theta$$

$$\phi_3-\phi_1 = \frac{4\pi a}{\lambda}\sin\theta$$

$$\phi_3-\phi_2 = \frac{2\pi a}{\lambda}\sin\theta$$

$$\therefore I_{total}(\theta) = I_1(\theta)\Big[3 + 4\cos\Big(\frac{2\pi a}{\lambda}\sin\theta\Big) + 2\cos\Big(\frac{4\pi a}{\lambda}\sin\theta\Big)\Big]$$

※

3-(3) Treating these as a $N=3$ grating,

$$I_{total}(\theta) = I_1(\theta)\cdot\frac{\sin^2\left(\frac{3\pi a}{\lambda}\sin\theta\right)}{\sin^2\left(\frac{\pi a}{\lambda}\sin\theta\right)}$$

Since

$$\sin^2\left(\frac{3\pi a}{\lambda}\sin\theta\right) = \sin^2(3\beta)$$

$$= \left(\sin\beta\cos 2\beta + 2\sin\beta\cos^2\beta\right)^2$$

$$= \sin^2\beta\left[2\cos 2\beta + 1\right]^2$$

$$= \sin^2\beta\left[1 + 4\cos 2\beta + 4\cos^2 2\beta\right]$$

$$= \sin^2\beta\left[3 + 4\cos 2\beta + 2\cos 4\beta\right]$$

$$= \sin^2\left(\frac{\pi a}{\lambda}\sin\theta\right)\left[3 + 4\cos\left(\frac{2\pi a}{\lambda}\sin\theta\right) + 2\cos\left(\frac{4\pi a}{\lambda}\sin\theta\right)\right]$$

$$\therefore\ I_{total} = I_1(\theta)\cdot\left[3 + 4\cos\left(\frac{2\pi a}{\lambda}\sin\theta\right) + 2\cos\left(\frac{4\pi a}{\lambda}\sin\theta\right)\right]$$

Same as the answer in Part (2)

4-(1) The reflectance viewing from the side is given by

$$R_{oil} = \frac{(n_g - n_{oil})^2}{(n_g + n_{oil})^2}$$

$$R_{water} = \frac{(n_g - n_w)^2}{(n_g + n_w)}.$$

Since $n_{oil} \neq n_w$, the amount of the reflected light seeing by eye is different, thus one tells the existence of a boundary separating the water and the oil.

4-(2) Again when viewing into the cup, in addition to the reflecting at air - oil interface, there is a finite reflect- at oil - water interface.

$$R = \frac{(n_{oil} - n_w)^2}{(n_{oil} + n_w)^2}.$$

From this reflection one can tell a layer of oil is floating on top of water.

4-(3) θ_B (from air) $= \tan^{-1}\dfrac{n_{oil}}{n_{air}} = 55.8^\circ$

4-(4) The Brewster angle from oil into water is

$$\theta_B(\text{oil-water}) = \tan^{-1}\frac{n_{water}}{n_{oil}} = 42.1^\circ$$

From Snell's law, the incident angle in the air θ_{air} that produces a refraction angle θ_B in oil $= 42.1^\circ$ is found as

$$n_{air}\sin\theta_{air} = n_{oil}\sin\theta_B(\text{oil})$$

$$\therefore \sin\theta_{air} = n_{oil}\sin\theta_B(\text{oil})$$

$$\theta_{air} = \sin^{-1}\left(n_{oil}\sin\theta_B(\text{oil})\right) = 80.5^\circ$$

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4-(5) The critical angle in oil beyond which the light is totally reflected at oil-water interface is

$$\theta_c(\text{oil}) = \sin^{-1}\left(\frac{n_w}{n_{oil}}\right) = 64.8^\circ$$

But the largest refraction angle θ_{oil} when the light is incident from the air is ($n_{air}\sin\theta_{air} = n_{oil}\sin\theta_{oil}$)

$$\theta_{oil} = \sin^{-1}\left(\frac{n_{air}\sin\theta_{air,max}}{n_{oil}}\right)$$

$$= \sin^{-1}\left(\frac{1}{n_{oil}}\right)$$

$$= 42.9^\circ$$

So we will never get to $\theta_c(\text{oil}) = 64.8^\circ$ #

5-(i) $\begin{bmatrix} -3 \\ 1 \end{bmatrix} = \sqrt{10} \cdot \begin{pmatrix} -3/\sqrt{10} \\ 1/\sqrt{10} \end{pmatrix}$

A linearly polarized light with a tilt angle angle

$$\alpha = -\tan^{-1}\left(\frac{1}{3}\right) = -18.4^\circ$$

from x-axis.

$$\begin{bmatrix} 2i \\ -i \end{bmatrix} = \left(\sqrt{5} \cdot i\right) \cdot \begin{pmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \end{pmatrix}$$

is a linearly polarized light with a tilt angle

$$\alpha = \tan^{-1}\left(\frac{-1}{2}\right) = -26.6^\circ$$

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$$\begin{bmatrix} 1-i \\ i-1 \end{bmatrix} = (1-i)\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \left(\sqrt{2}(1-i)\right)\begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$$

is again a linearly polarized light with a tilt angle from x-axis

$$\alpha = \tan^{-1}\left(\frac{-1}{1}\right) = -45^\circ$$

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5-(2) $\vec{E}(t) = 3\cos(\pi/4 - \omega t)\hat{x} - 4\cos\omega t \cdot \hat{y}$

$$= 3\cos(\pi/4 - \omega t)\hat{x} + 4\cos(\pi - \omega t)\cdot\hat{y}$$

$$\therefore \tilde{E} = \begin{pmatrix} 3e^{i\pi/4} \\ 4e^{i\pi} \end{pmatrix} = \left(5e^{i\pi/4}\right)\begin{pmatrix} 3/5 \\ (4/5)\, e^{i3\pi/4} \end{pmatrix}$$

is an elliptically polarized light, with its major axes not aligned with x-y axis since $\phi_y - \phi_x \neq \pm \pi/2$.

5-(3) $\vec{E}(t) = 2\cos(\pi/4 + \omega t)\hat{x} + 2\sin(\omega t - \pi/4)\hat{y}$

$$= 2\cos(-\pi/4 - \omega t)\hat{x} + 2\cos\left(\frac{\pi}{2} - (\omega t - \pi/4)\right)\hat{y}$$

$$= 2\cos(-\pi/4 - \omega t)\hat{x} + 2\cos\left(\frac{3\pi}{4} - \omega t\right)\hat{y}$$

$$\therefore \tilde{E} = \begin{pmatrix} 2e^{-i\pi/4} \\ 2e^{i3\pi/4} \end{pmatrix} = \left(2\sqrt{2}\, e^{-i\pi/4}\right)\begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$$

It is a linearly polarized light with a tilt angle from x-axis

$$\alpha = \tan^{-1}(-1/1) = -45^\circ$$

6-(1) One can simply use a linear polarizer with TA parallel to x-axis,

$$M_{pL}(TA \parallel \hat{x}) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\tilde{E}_{out} = M_{pL}(TA \parallel \hat{x})\, \tilde{E}_{inc}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$ #

6-(2) Method #1: HWP

One can use a half-wave plate with its FA set at $\theta = 3\pi/8$

$$M_{HWP}(FA @ \theta = 3\pi/8 \text{ fm } x) = \begin{pmatrix} \cos 3\pi/4 & \sin 3\pi/4 \\ \sin 3\pi/4 & -\cos 3\pi/4 \end{pmatrix}$$

$$= \begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

$$\therefore \tilde{E}_{out} = M_{HWP}\, \tilde{E}_{inc} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$ #

Method #2: Rotator that add $\alpha = +45^\circ$

$$M_{rotator}(\alpha = 45^\circ) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$\tilde{\hat{E}}_{out} = M_{rotator}(\alpha = 45^\circ)\,\tilde{\hat{E}}_{inc} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

6-(3) Method #1. QWP

One can use a QWP with FA ∥ $\hat{y}$ axis

$$M_{QWP}(FA \parallel \hat{y}) = \begin{pmatrix} e^{i\pi/2} & 0 \\ 0 & 1 \end{pmatrix} = e^{i\pi/2}\begin{pmatrix} 1 & 0 \\ 0 & e^{-i\pi/2} \end{pmatrix}$$

$$\tilde{\hat{E}}_{out} = M_{QWP}\,\tilde{\hat{E}}_{inc} = \left(e^{i\pi/2}\right)\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix} = \tilde{\hat{E}}_R$$

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Method #2. QWP + HWP.

One can use a QWP with FA ∥ $\hat{x}$, followed by a HWP with FA ∥ $\hat{x}$ axis, thus

$$M_{QWP}(\text{FA} \parallel \hat{x}) = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$$

$$M_{HWP}(\text{FA} \parallel \hat{x}) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\tilde{\hat{E}}_{out} = M_{HWP} \cdot M_{QWP} \tilde{\hat{E}}_{inc}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix} = \tilde{\hat{E}}_R .$$

6-(4) See the unused method in Part (2)

6.(5) See the unused method in Part (3)