Solution to Physics 9C-A Final (2016)

1-(a) Electric field at A is the linear superposition of the electric field produced by the positively charged shell at A and the field produced by the negative charged shell.

But the field by the positively charged shell at its center is zero. Thus

$$\vec{E}_A = \vec{E}_A \Big|_{\substack{\text{negatively} \\ \text{charged shell}}} = \frac{1}{4\pi\varepsilon_0} \frac{(-Q)}{L^2} \cdot (-\hat{x})$$

$$= \frac{1}{4\pi\varepsilon_0} \frac{Q}{L^2} \hat{x}$$

1-(b) Electric field at C is the linear superposition of the field produced by the positively charged shell at C and the field produced by the negatively charged shell at C:

$$\vec{E}_C = \vec{E}_C \Big|_{\substack{\text{positively} \\ \text{charged} \\ \text{shell}}} + \vec{E}_C \Big|_{\substack{\text{negatively} \\ \text{charged} \\ \text{shell}}}$$

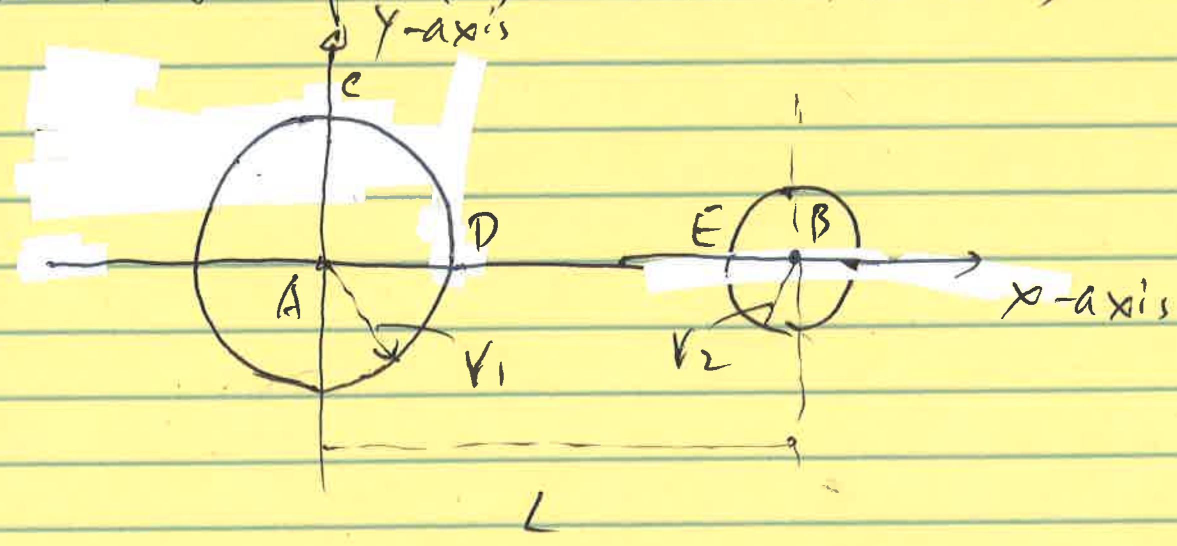
$$= \frac{1}{4\pi\varepsilon_0}\frac{Q}{r_1^2}\hat{y} + \frac{1}{4\pi\varepsilon_0}\frac{(-Q)}{L^2+r_1^2}\cdot\frac{(-L)\hat{x}+r_1\hat{y}}{\sqrt{L^2+r_1^2}}$$

$$= \frac{Q}{4\pi\varepsilon_0}\left(\frac{L}{(L^2+r_1^2)^{3/2}}\right)\hat{x} + \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{r_1^2} - \frac{r_1}{(L^2+r_1^2)^{3/2}}\right)\hat{y}$$

1-(c) The electric potential difference $V_A - V_B$ is the linear superposition of $(V_A - V_B)$ due to the positively charged shell and $(V_A - V_B)$ due to the negatively charged shell.

$$V_A - V_B = \int_A^B \vec{E}\cdot d\vec{l} = \int_A^B \vec{E}(+Q)\cdot d\vec{l} + \int_A^B \vec{E}(-Q)\cdot d\vec{l}$$

Choose the path from A to B along x-axis.



$$\int_A^B \vec{E}(+Q)\cdot d\vec{l} = \underbrace{\int_A^D \vec{E}(+Q)\cdot d\vec{l}}_{0} + \int_D^B \vec{E}(+Q)\cdot d\vec{l}$$

$\vec{E}(+Q) = 0$ inside r_1

$$= \frac{Q}{4\pi\varepsilon_0}\cdot\left(\frac{1}{r_1} - \frac{1}{L-r_1}\right)$$

$$\int_A^B \vec{E}(-Q)\cdot d\vec{l} = \int_A^E \vec{E}(-Q)\cdot d\vec{l} + \underbrace{\int_E^B \vec{E}(-Q)\cdot d\vec{l}}_{0}$$

$\vec{E}(-Q) = 0$ inside r_2

$$= \frac{(-Q)}{4\pi\varepsilon_0}\cdot\left(\frac{1}{L-r_2} - \frac{1}{r_2}\right)$$

$$= \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{r_2} - \frac{1}{L-r_2}\right)$$

$$\therefore\ V_A - V_B = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{r_1} + \frac{1}{r_2} - \frac{1}{L-r_1} - \frac{1}{L-r_2}\right)$$

2-(a) The charges on both capacitors are the same as they are connected in series.

The potential drop the two capacitors is the same as V_{ab} as after V_{ab} is applied for a long time, there should be no current flowing through R_3. The charge on the two capacitors is related to the potential drop V_{ab} across them by the network capacitance

$$C_{12} = \frac{C_1 \cdot C_2}{C_1 + C_2}$$

Thus

$$Q_{10} = Q_{20} = V_{ab} \cdot C_{12} = \frac{V_{ab} \cdot C_1 \cdot C_2}{C_1 + C_2}$$

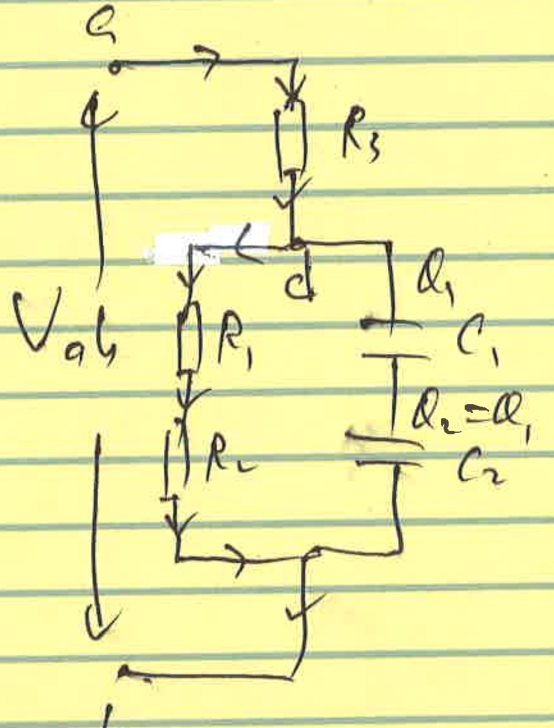
2-(b) After the switch is closed for a long time, no more current flows to the arm of capacitors. As a result, the current through all three resistors is the same,

$$I = \frac{V_{ab}}{R_1 + R_2 + R_3}$$

Now the potential drop across the two capacitors connected in series is

$$V_{db} = I \cdot (R_1 + R_2)$$

$$= \frac{V_{ab} \cdot (R_1 + R_2)}{R_1 + R_2 + R_3}.$$



The charges on both the capacitors are the same as they are in series,

$$Q_1 = Q_2 = V_{db} \cdot C_{12}$$

$$= \frac{V_{ab} \cdot (R_1 + R_2)}{R_1 + R_2 + R_3} \cdot \frac{C_1 \cdot C_2}{C_1 + C_2}$$

2-(c) Immediately after the switch is closed, the charge on both C_1 and C_2 cannot change immediately, thus the potential drop across them remain unchanged:

$$V_{db} = \frac{Q_{10}}{C_1} + \frac{Q_{20}}{C_2} = V_{ab}$$

As a result, there is no current through R_3 since $V_{ab} = V_{db} = V_{ad} + V_{db}$

$$\therefore V_{ad} \text{ (across } R_3) = 0$$

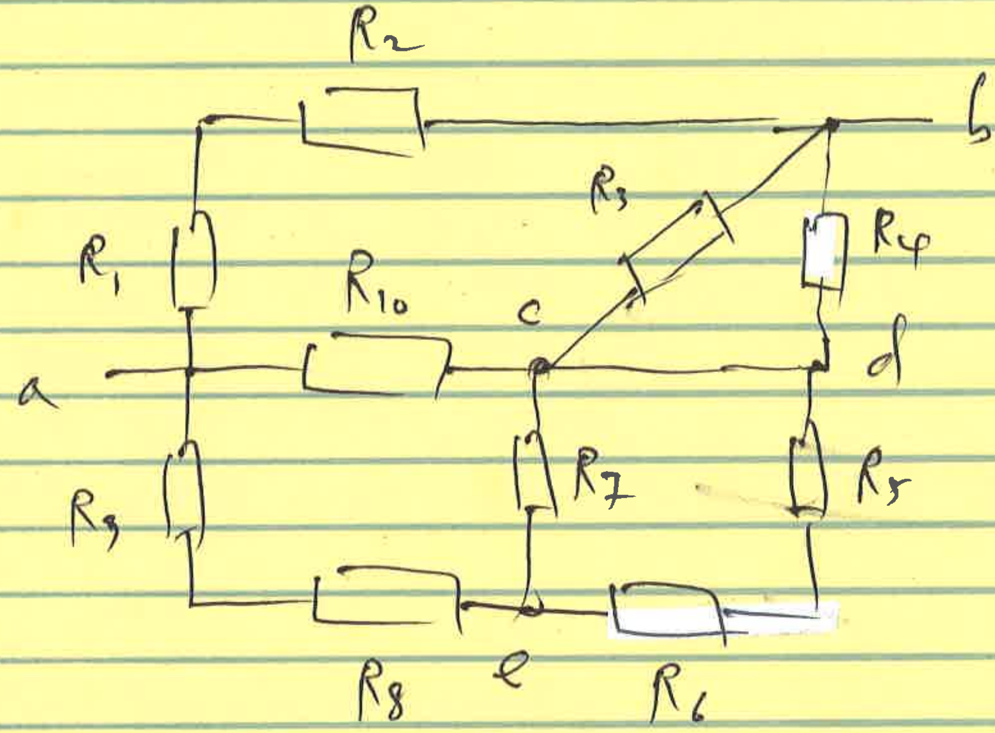
By Ohm's law, $I_3 = 0$.

Since $V_{db} = V_{ab}$ at that moment, by Ohm's law, the current through both R_1 and R_2 are equal and given by

$$I_1 = I_2 = \frac{V_{ab}}{R_1 + R_2}$$

$\cancel{\times}$

3-(a)



$$R_{12} = R_1 + R_2$$

$$R_{34} = \frac{R_3 \cdot R_4}{R_3 + R_4}$$

$$R_{56} = R_5 + R_6$$

$$R_{567} = \frac{R_7 \cdot R_{56}}{R_7 + R_{56}}$$

$$R_{89} = R_8 + R_9$$

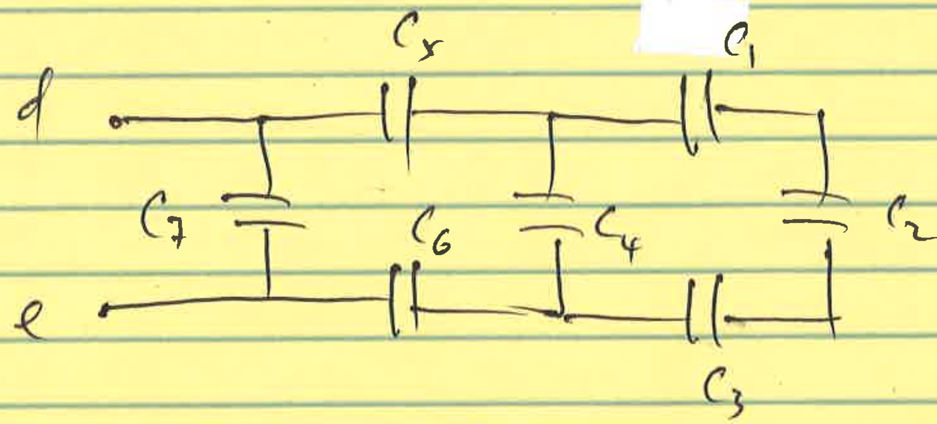
$$R_{56789} = R_{89} + R_{567}$$

$$R_{5678910} = \frac{R_{56789} \cdot R_{10}}{R_{10} + R_{56789}}$$

$$R_{345678910} = R_{34} + R_{5678910}$$

$$\therefore R_{ab} = \frac{R_{12}\cdot R_{345678910}}{R_{12} + R_{345678910}}$$

3-(b)



$$C_1 = C_2 = C_3 = C_4 = C_5 = C_6 = C_7 = c$$

$$\frac{1}{C_{123}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} = \frac{3}{c}$$

$$\therefore C_{123} = c/3$$

$$C_{1234} = C_{123} + C_4 = \frac{c}{3} + c = \left(4/3\right)\cdot c$$

$$\frac{1}{C_{123456}} = \frac{1}{C_5} + \frac{1}{C_6} + \frac{1}{C_{1234}} = \frac{1}{c} + \frac{1}{c} + \frac{3}{4c} = \frac{11}{4c}$$

$$\therefore C_{123456} = \frac{4}{11}c \implies C_{de} = C_{1234567} = C_7 + C_{123456} = \frac{15}{11}c$$ ✱

4-(a) The magnetic field produced by the current loop at z>0 is

$$\vec{B}(z) = (-\hat{z}) \cdot \frac{\mu_0 \cdot I \cdot R^2}{2(R^2+z^2)^{3/2}}$$

The torque experienced by $\vec{m}$ at this point is

$$\vec{\tau}_m = \vec{m} \times \vec{B}$$

$$= (m\hat{x}) \times (-\hat{z}) \cdot \frac{\mu_0 \cdot I \cdot R^2}{2(R^2+z^2)^{3/2}}$$

$$= \hat{y} \cdot \frac{m \cdot \mu_0 \cdot I \cdot R^2}{2 \cdot (R^2+z^2)^{3/2}}$$

4-(b) The current loop represents a permanent magnet with the south pole pointing along the positive z-axis. m is also a permanent magnet with the south pole facing the current loop. Since like poles repel each other, the force on m is along the positive z-axis, away from the current loop.*

4-1(c) The force on the permanent magnet $\vec{m}$ is the negative gradient of its potential energy along the z-axis.

$$U(z) = -\vec{m}\cdot\vec{B}$$

$$= -(m\hat{z})\cdot\left(-\hat{z}\,\frac{\mu_0 I\cdot R^2}{2(R^2+z^2)^{3/2}}\right)$$

$$= \frac{m\cdot\mu_0\cdot I\cdot R^2}{2(R^2+z^2)^{3/2}}$$

$$\vec{F}_{\text{on } \vec{m}} = -\left(\frac{dU(z)}{dz}\right)\cdot\hat{z}$$

$$= (-)\left(-\frac{3}{2}\right)\cdot(2z)\cdot\frac{\mu_0\cdot I\cdot R^2}{2(R^2+z^2)^{5/2}}\,\hat{z}$$

$$= +\frac{3\cdot\mu_0\cdot I\cdot R^2\cdot z}{2(R^2+z^2)^{5/2}}\cdot\hat{z}$$

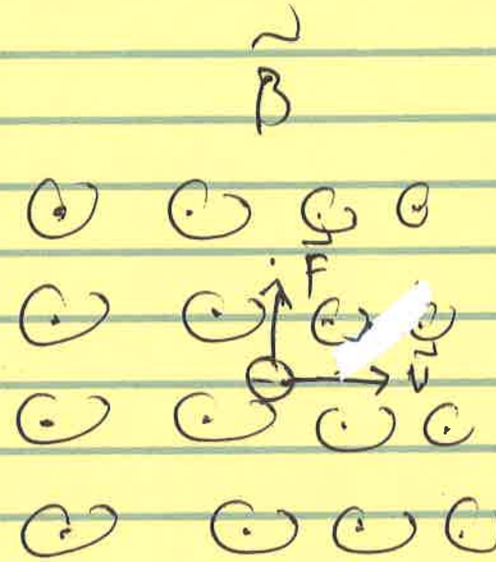
For $z>0$, $\vec{F}$ is along positive z, repulsion

For $z<0$, $\vec{F}$ is along negative z, also repulsion

5-(a) Two methods.

Method #1:

Let the electron be moving to the right as indicated.



The force and thus the acceleration on the electron is

$$\tilde{F}_{on\, q_e} = (-e)\cdot \tilde{v}\times\tilde{B}$$

pointing upward. Thus the cyclotron motion of the electron is counter-clockwise.

Method #2:

From

$$m\frac{d\tilde{v}}{dt} = q_e \tilde{v}\times\tilde{B}$$

$$\Rightarrow \frac{d\tilde{v}}{dt} = \left(-\frac{q_e\tilde{B}}{m}\right)\times\tilde{v}$$

It describes a circular motion with the angular velocity $\tilde{\omega} = -q_e\tilde{B}/m = +\frac{e}{m}\tilde{B}$: CCW

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5-(b) $R_c = \dfrac{m \cdot v}{|q_e| \cdot B} = \dfrac{9.1\times10^{-31}\,kg \times 4\times10^{7}\,m/sec}{1.6\times10^{-19}\,C \times 0.2\,T}$

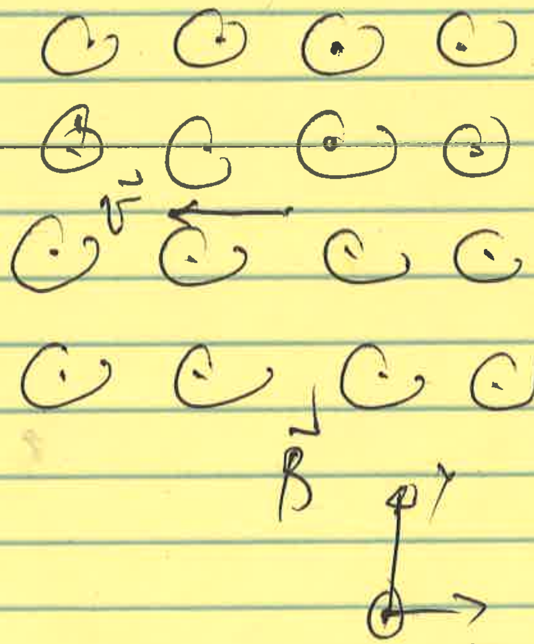
$= 1.14\times10^{-3}\,m = 1.14\,mm$ ✱

5-(c) When the electron is moving to the left, the applied electric $\vec{E}$ needs to be such that the total force on the electron is zero so that it will continue moving at $\vec{v}$ to the left.

$$\vec{F}_{on\,q_e} = q_e\left(\vec{E} + \vec{v}\times\vec{B}\right) = 0$$

$$\therefore \vec{E} = -\vec{v}\times\vec{B}$$

$$= (-\hat{y})\,vB$$



or $\vec{E} = -\vec{v}\times\vec{B}$

$$= -\left(v(-\hat{x})\right)\times B\hat{z}$$

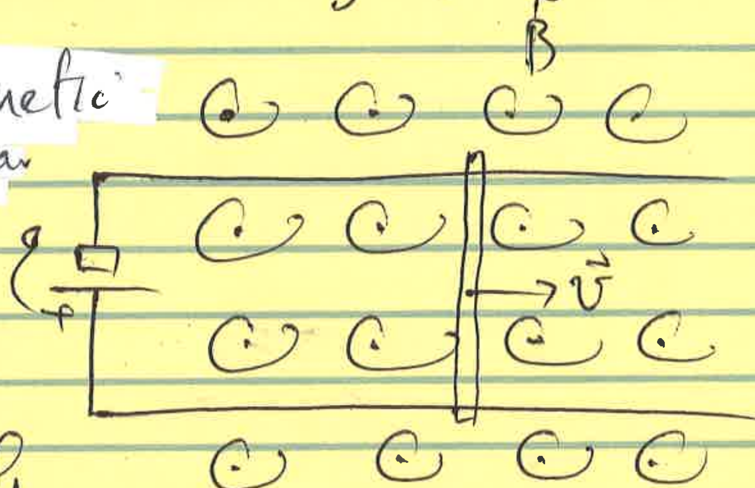
$$= vB\,\hat{x}\times\hat{z}$$

$$= v\cdot B\cdot(-\hat{y})$$ ✱

6-(a) If we want the bar to move to the right the magnetic force on the bar when it carries a current produced by $\mathcal{E}$ has to be pointing to the right.

Since the magnetic force on the bar is

$$\vec{F}^{(M)} = I\vec{l} \times \vec{B}$$

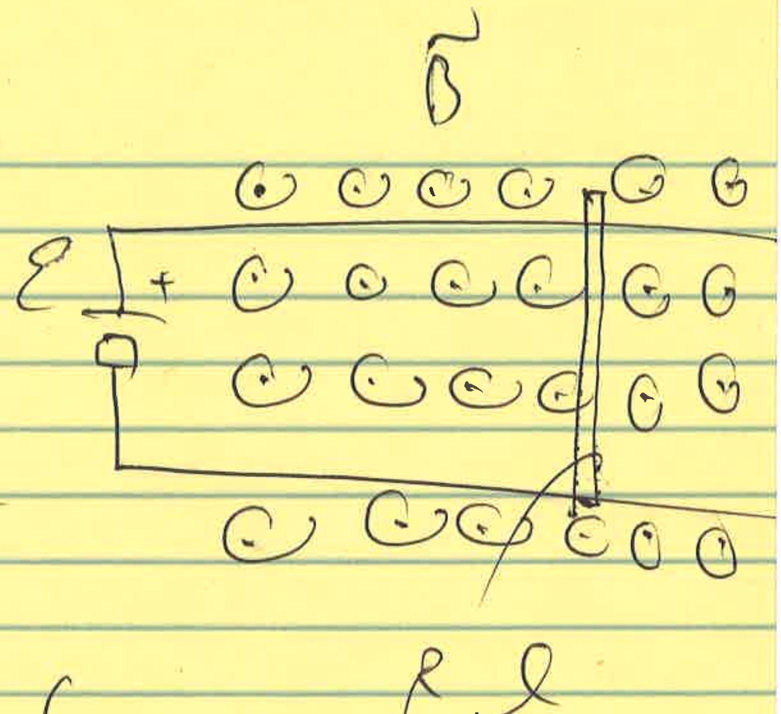


with $\vec{l}$ along the direction of the current. For $F^{(M)}$ to point to the right the current must flow upward through l. So the battery $\mathcal{E}$ is connected downward.

The magnitude of the battery has to be such that it cancels out the motion emf $\mathcal{E} = vlB$ at the desired velocity v and there is no more current flowing through the bar and thus no more magnetic force to accelerate it further.

$$\therefore \mathcal{E}_{battery} = v \cdot l \cdot B$$

6—(B) If $\mathcal{E}$ is connected upward, it produces a clockwise current



$$I\Big|_{\substack{\text{due to}\\ \mathcal{E}}} = \frac{\mathcal{E}}{R}.$$

As a result, the bar experiences a magnetic force

$$\vec{F}^{(m)} = I\vec{l} \times \vec{B}$$
$$= I l \cdot B \cdot (-\hat{y}) \times \hat{z}$$
$$= I \cdot l \cdot B \cdot (-\hat{x})$$

We need an external force $\vec{F}_{ext}$ so that

$$\vec{F}_{ext} + \vec{F}^{(m)} = 0$$

$$\therefore \vec{F}_{ext} = -\vec{F}^{(m)} = I \cdot l \cdot B \cdot \hat{x} = \left(\frac{\mathcal{E}}{R}\right) \cdot l \cdot B \cdot \hat{x}$$

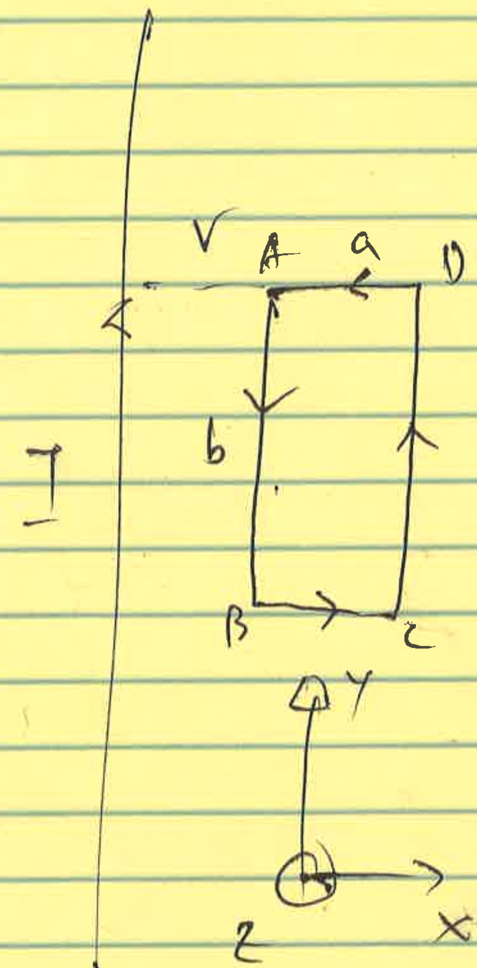
along the positive x (to the right)

7-(a) The magnetic forces on the five sides of a cancel each other.

The magnetic field due to the long wire at AB segment is

$$\tilde{B}(r) = \frac{\mu_0 I}{2\pi r}(-\hat{z})$$

Thus the magnetic force on AB segment with I_2 flowing from A to B



$$\vec{F}_{on\,AB} = I_2 \vec{AB} \times \tilde{B}(r)$$

$$= \frac{\mu_0 I \cdot I_2 \cdot b}{2\pi r}(-\hat{y}) \times (-\hat{z})$$

$$= \frac{\mu_0 I I_2 \cdot b}{2\pi r}\hat{x}$$

The magnetic field due to I at CD segment

$$\vec{B}(r+a) = \frac{\mu_0 I}{2\pi(r+a)}(-\hat{z})$$

The force on CD segment with I_2 flowing upward from C to D

$$\vec{F}_{\text{on } CD} = I_2 \overrightarrow{CD} \times \vec{B}(r+a)$$

$$= \frac{\mu_0 I \cdot I_2 \cdot b}{2\pi(r+a)} (\hat{y}) \times (-\hat{z})$$

$$= -\frac{\mu_0 I \cdot I_2 \cdot b}{2\pi(r+a)} \hat{x} .$$

The total force

$$\vec{F}_{\text{on the loop}} = \vec{F}_{\text{on } AB} + \vec{F}_{\text{on } CD}$$

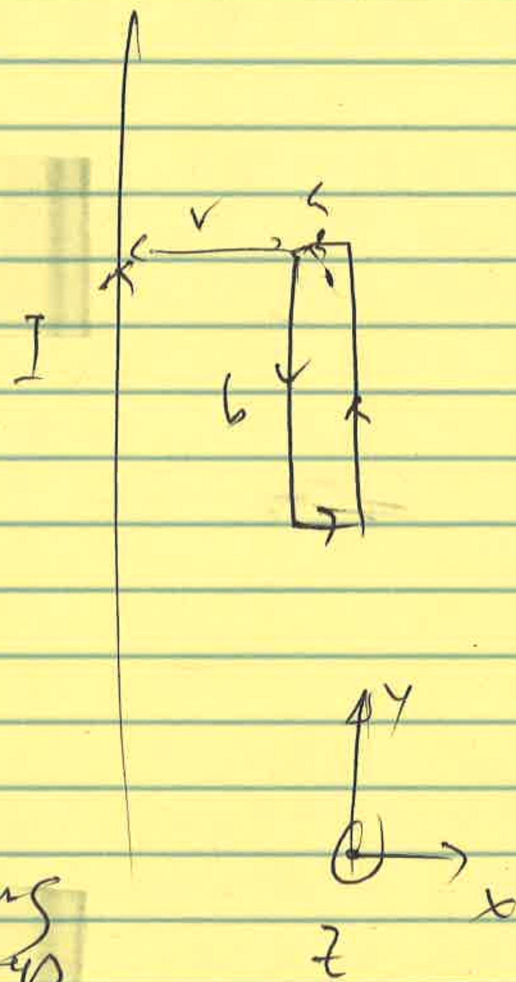
$$= \frac{\mu_0 I \cdot I_2 \cdot a \cdot b}{2\pi r (r+a)} \hat{x}$$

$$\cong \frac{\mu_0 I \cdot (I_2 a b)}{2\pi r^2} \hat{x}$$

away from the loop above.

7-(b) Since $a \ll r$, the magnetic field due to I through the loop

$$\vec{B}(r) = -\frac{\mu_0 I(t)}{2\pi r}(\hat{z})$$



Thus the magnetic flux through a counter-clockwise loop is

$$\Phi_B = \vec{B}(\vec{r})(ab\,\hat{z})$$

$$= -\frac{\mu_0 ab}{2\pi r} I(t)$$

The induced emf along the counter-clockwise loop is

$$\mathcal{E}_{ind} = \oint \vec{E}\cdot d\vec{l} = -\frac{d\Phi_B}{dt} = \frac{\mu_0 ab}{2\pi r}\cdot\alpha$$

So the induced current is also counter-clockwise

$$I_{loop} = \frac{\mathcal{E}_{ind}}{R} = \frac{\mu_0 ab\cdot\alpha}{2\pi r\cdot R}$$