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A FURTHER EXAMINATION OF OPTICAL ANALOGY OF QUANTUM TRANSPORT IN MESOSCOPIC STRUCTURES

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We show that the transmittance of light through waveguides is analogous to the conductance of electrons through mesoscopic wires *only* when a quasi-monochromatic light source is used. We also show that when casting the Maxwell equations into a Schrödinger-type equation in an optical waveguide with a spatially varying dielectric function $\varepsilon(\omega,r)$ (the optical analog of a mesoscopic electron conductor with disorders), a velocity-like, non-hermitian term in the *effective* Hamiltonian prevails. The latter distinguishes the optical case from the electron case, in addition to the vector nature of electromagnetic waves.

Quantum transport of electrons in solids has attracted considerable attention in the past decade.[1-6] The subject is closely linked to a number of most interesting problems in condensed matter physics such as electron localization in disordered solids and conductance fluctuations in mesoscopic conductors. The problems have been successfully treated with quantum transport theories of Landauer-type by a number of authors.[2,4,6,7] Similar approaches have also been applied to several analogous cases involving electromagnetic waves.[6,8,9,10]

Experimentally, van Wees,[11] Wharam,[12] Faist[13] and their coworkers recently observed quantized electron conductance through point contacts or electron waveguides. These new experiments have stimulated the research on ballistic and partially diffusive transport of electrons.[11,14,15,16] The optical analogy of ballistic transport of electrons using a monochromatic light has also been discussed by Housten and Beenakker.[17] The experimental demonstration of quantized transmittance of diffusive light through a narrow slit has been elegantly performed by Montie and coworkers.[18] It is believed that the studies of electron transport in the mesoscopic limit should be benefited from a thorough study of the optical transmittance through waveguides for two reasons. Theoretically, electromagnetic waves represent simple non-interacting systems which are readily subject to thorough theoretical investigations. Experimentally, optical waveguides can be prepared relatively easily.

In this communication, we shall address two issues regarding the optical analogy of electron transport in mesoscopic conductors. The first concerns with the physical grounds on which the optical transmittance through an optical waveguide is considered analogous to the electron conductance through a mesoscopic wire. A clarification of this issue is necessary because the electron conductance is often pictured as the response of a wire to an external applied field while there is no external field involved in the case of the optical transmittance. We then show that only the transmittance of a *quasi-monochromatic light* with a finite frequency bandwidth $\Delta\omega$ through a waveguide is exactly analogous to the conductance of electrons in a mesoscopic conductor. The second issue concerns with the difference between the electron conductance and the optical transmittance arising from that the former is described by the Schrödinger equation while the latter by Maxwell equations. As we will show, when casting the Maxwell equations in a waveguide with a spatially varying dielectric function $\varepsilon(\omega,r)$ into a vector form of a Schrödinger eigenvalue equation, a scattering potential term, $-\nabla(\nabla\cdot E)$, makes the effective Hamiltonian *non-hermitian*. This makes the optical transmittance an interesting case in its own right.

We first consider the optical transmittance through one section L of a long waveguide L_0 with a cross section A. L_0 is used for the quantization along the long waveguide. The section L is far away from either ends of the waveguide, namely, $L << L_0$. There can be dielectric impurities inside the section characterized by a spatially varying dielectric constant $\varepsilon(\omega,r)$, but none outside the section throughout the waveguide. Such a "disordered" section can be viewed as a piece of "optical mesoscopic wire". Let a set of *monochromatic* light waves at ω_0 incident at the left end of the "disordered" section. The sum of the *total fluxes* coming out of the right end may be defined as the optical transmittance of the section,

$$I_{optl}(\omega_0) = \sum_{a,b} n_a(\omega_0)v_a(\omega_0)T_{ab,optl}(\omega_0). \tag{1}$$

Here, a and b denote the incident channel and the transmitted channel, respectively. They include the polarization labels as appropriate. $n_a(\omega_0)$ is the photon numbers per unit length along the waveguide at ω_0, and $v_a(\omega_0)$ is the corresponding group velocity along the waveguide. $T_{ab,optl}(\omega_0)$ is the transmission probability from an incident channel a into an outgoing channel b. Eq. (1), valid for a monochromatic optical wave, is not yet analogous to the conductance of electrons through a mesoscopic wire as will be clear shortly. Note that $I_{optl}(\omega_0)$ depends explicitly upon the group velocities of each open channel. This point has been previously overlooked by other authors.[17] *If, instead, a set of quasi-monochromatic waves with a center frequency ω_0 and a bandwidth $\Delta\omega$ are incident upon the section, the total transmittance I_{optl} can now be shown directly analogous to the electron conductance.* We assume that $n_a(\omega)$ does not vary over the bandwidth $\Delta\omega$ such that $n_a(\omega) \approx n_a(\omega_0)$. For each incident channel a, one must sum over all possible k_{az} states within $\Delta\omega$ and thus involve an integration of the one dimensional density of states $D_a(\omega) = (L_0/2\pi)(dk_{az}/d\omega) = (L_0/2\pi)[1/v_a(\omega)]$. k_{az} is the wave vector component along the waveguide. The total transmittance is obtained by integrating $I_{optl}(\omega)$ given in Eq. (1) over $\Delta\omega$,

$$I_{optl} = \int_{\omega_0}^{\omega_0+\Delta\omega} D(\omega) I_{optl}(\omega) d\omega = \frac{\Delta\omega}{2\pi}\sum_{a,b} N_a(\omega_0) T_{ab,optl}(\omega_0). \tag{2}$$

$N_a(\omega_0) = n_a(\omega_0)L_0$ is the total number of photons in the incident channel a.

To see the analogy, we consider the electron conductance through a mesoscopic wire sandwiched between two much longer and perfect conductors with the same cross section. Following an argument by Anderson and coworkers,[2] when there is no electric field applied across a wire, we have an equal electron flux impinging on both ends of the wire so that the net current is zero. To obtain the electron "transmittance" or the conductance, we need to apply a voltage ΔV to create an imbalance of the chemical potential $e\Delta V$ across the mesoscopic wire. This leads to a population difference between two ends over an energy bandwidth $e\Delta V$ at the Fermi level E_f, $\Delta n_a(E) = [dn_a(E)/dE]_{E=E_f}(e\Delta V)$. The transmitted flux is now dependent upon the voltage ΔV. Here, $n_a(E)$ is the number of electrons per unit length along the wire in channel a with energies less than E. It is natural to define the conductance Γ as the total transmitted flux I divided by the voltage ΔV or the bandwidth $e\Delta V/\hbar$. Clearly, we expect Γ to have the same form as for I_{optl} except for the unit e^2/h,

$$\begin{aligned}\Gamma = \frac{I}{\Delta V} &= \sum_{a,b} \frac{n_a(E_f+e\Delta V) - n_a(E_f)}{\Delta V} e v_a T_{ab,electron} \\ &= \frac{1}{\Delta V}\sum_{ab}\left[\frac{g_c}{2\pi}\int_{E_f}^{E_f+e\Delta V}\frac{dE}{\hbar v_a}\right] e v_a T_{ab,electron} \\ &= \frac{e^2}{h}\sum_{a,b} g_c T_{ab,electron}.\end{aligned} \tag{3}$$

$T_{ab,electron}$ is the transmission probability of an electron wave at the Fermi energy E_f from an incident channel a into an outgoing channel b. $v_a = (1/\hbar)dE/dk_{az}$ is the longitudinal group velocity. g_c is the degeneracy factor of an allowed energy level in the conductor and is equal to 2 at the equilibrium as required by Pauli's principle. In the electron case, the effect of the density of states is already included when a finite voltage ΔV is applied and the equivalent bandwidth is given by $e\Delta V/\hbar$.

Equations (2) and (3) are identical except for the quantum statistical requirements on the degeneracy factor in each open channel. For electromagnetic waves, the degeneracy factor $N_a(\omega_0)$ in a channel a is not limited. Eq. (3) is known as the Landauer formula for the electron conductance through mesoscopic wires and is now extended to electromagnetic waves in wave guides. To evaluate the transmission probability T_{ab}, we generally resort to the Schrödinger equation for electrons and Maxwell equations for optical waves.[4]

Now we are led to the second issue. The optical analogy can be further elucidated by mapping Maxwell equations onto a vector form Schrödinger equation. We rewrite Maxwell equations in a wave guide section *with a spatially varying dielectric function* $\varepsilon(\omega,r)$ in the following way. We use the electric field $\mathbf{E}$ as the "equivalent wave function", because in the region away from the "disordered" section, the photon density is simply proportional to $|E|^2$ and the photon flux is given by the product of the group velocity v and $|E|^2$. By introducing $\varepsilon(\omega)$ as the asymptotic value of $\varepsilon(\omega,r)$ away from the "disordered" section and $\delta\varepsilon(r) \equiv \varepsilon(\omega) - \varepsilon(\omega,r)$, we can define a second-rank tensor, $V_{ij}(r)$, as *the effective dielectric scattering potential*,

$$V_{ij}(r) \equiv \frac{\delta\varepsilon(r)\omega^2}{c^2}\delta_{ij} - \frac{\partial^2 \ln[\varepsilon(\omega,r)]}{\partial x_j \partial x_i} - \frac{\partial \ln[\varepsilon(\omega,r)]}{\partial x_j}\frac{\partial}{\partial x_i}, \tag{4}$$

Now by using an effective Hamiltonian $H_{ij} = -\delta_{ij}\Delta + V_{ij}(r)$, and an effective eigenvalue $W = \varepsilon(\omega)\omega^2/c^2$, we may cast Maxwell equations in a matrix form analogous to the Schrödinger equation,

$$\overleftrightarrow{H}\cdot\vec{E} = W\vec{E}. \tag{5}$$

To calculate the transmission probability $T_{ab,optl}(\omega_0)$, we generally apply the quantum mechanical theory of scattering.[4,19] In the absence of dielectric scattering potential, $V_{ij}(r) = 0$, we find $T_{ab,optl} = \delta_{ab}$, and from Eq. (2), $I_{optl} = (\Delta\omega/2\pi)\Sigma_a N_a(w_0)$. The sum is over all the open (propagating) channels in the waveguide for a fixed incident photon energy $\hbar\omega_0$. If one changes the cross section A of the waveguide or equivalently the photon energy $\hbar\omega_0$, more or fewer channels will be open at discrete intervals of A/λ^2 ($\lambda = 2\pi c/\omega_0$ is the vacuum wavelength of the incident light), and one should observe a step-like transmittance variation as a function of the photon energy.[11, 17] When $V_{ij}(r) \neq 0$, it is necessary to apply the general scattering theory.[4,19]

We now discuss the difference between the propagation of an electron wave and an electromagnetic wave. As one may have noticed, the effective Hamiltonian H_{ij} for an electromagnetic wave is not Hermitian as it contains *pure* derivatives, $\partial/\partial x_i$, acting upon the "wave function". This has important consequences which distinguish the optical transmittance from the electron conductance. First, the micro-reversibility of the transition matrix $\mathbf{T}$ is now broken so that $T_{ab,optl}(\omega_0) \neq T_{ba,optl}(\omega_0)$ even though the effective Hamiltonian has the time-reversal symmetry. This should be observable in an optical experiment. Secondly, one may no longer apply the fully retarded Green's function approach which requires the Hamiltonian to be Hermitian.[4] Finally, spatial eigenfunctions belonging to different eigenvalues are no longer necessarily orthogonal to each other. These consequences make the optical transmittance a rather interesting case in its own right.

There are, however, two interesting and important limiting cases which are worthy of consideration in more details. One limit is when the variation of the dielectric function $\varepsilon(\omega,r)$ is small over the scale of the wavelength so that we can neglect the derivative terms in the scattering potential in Eq. (4), then $V_{ij}(r) \approx [\delta\varepsilon(r)\omega^2/c^2]\delta_{ij}$. It is easy to see that the effective Hamiltonian H_{ij} becomes approximately Hermitian again. By extending to the vector case for electromagnetic waves, the retarded Green's function approach and the scattering theory as described by Fisher and Lee can now be extended to the calculation of the optical transmittance.[4,20] For example, if we define the unperturbed electric field $\mathbf{E}(\omega,\rho;a)exp(ik_{az}z)$ of an open channel a so that $\int d\rho dz |\mathbf{E}(\omega,\rho;a)|^2 = 1$ with the integration over the volume of the entire waveguide AL_0, and define the matrix elements of the Green's function $G^{(+)}(W) = (W + i\eta - H)^{-1}$ coupling an incoming channel a to an outgoing channel b as in Ref. 4,

$$G_{ab}^{(+)}(z,z') = \frac{\omega L_0}{\pi c^2} \iint_{(A)} d\rho d\rho' E(\omega,\rho;b) G^{(+)}(r,r') E(\omega,\rho';a), \tag{6}$$

the optical transmittance can be expressed as [4]

$$I_{optl} = \frac{\Delta\omega}{2\pi} \sum_{a,b} N_a(\omega) / G_{ba}^{(+)}(z\rightarrow+\infty, z'\rightarrow-\infty) /^2 v_a v_b. \tag{7}$$

The second limiting case is when the variation of $\varepsilon(\omega,r)$ is not small over the scale of the wavelength and yet the strength of the scattering potential $V_{ij}(r)$ is weak compared to the difference between the squares of the transverse wave vectors of two neighboring propagating channels. In this case, $T_{ab,optl}$ can be calculated using the Born approximation. By defining the matrix element of the scattering potential $V_{ij}(r)$ between the incident channel a and the reflection channel b

$$V_{ab} = \frac{iL_0}{2\pi} \int_{(A,L_0)} dr\, E(\omega,\rho;b) \tilde{V}(r) E(\omega,\rho;a)\, exp[i(k_{az}+k_{bz})z], \tag{8}$$

we find the reflection coefficient $r_{ab,optl} = V_{ab}c^2/(v_a v_b)^{1/2}\omega$, $T_{ab,optl} = 1 - \Sigma_b / r_{ab,optl} /^2$,[15,16] and finally

$$I_{optl} = \frac{\Delta\omega}{2\pi} \sum_a N_a(\omega) [1 - \sum_b / r_{ab,optl} /^2]. \tag{9}$$

A number of observations from $r_{ab,optl}$ and Eq. (9) are in order: (i) It is easily seen that whenever a new channel a' is just open, the group velocity $v_{a'}$ is small. Consequently, the corresponding $r_{ab,optl}$ is large. Then, one should observe a large reduction in I_{optl} near such an opening. (ii) If the scattering potential is point-like, the transmittances of *all incidence channels* will be reduced by an enhanced scattering into the backward direction of the new channel a' since a *δ-function* can couple any two waves with arbitrary difference in wave vector. When the number of the incidence channels is large, a much larger dip of the total transmittance should be expected. This was qualitatively observed by Faist and coworkers in the case of electron transport and later explained by Wang and Feng.[13,16]

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