Thermal Johnson Noise Generated by a Resistor

REFERENCES

- Reif, *Fundamentals of Statistical and Thermal Physics*, pp. 589 - 594
- Kittel, *Thermal Physics*, pp. 98-102
- Kittel, *Elementary Statistical Physics*, pp. 141-149.
- Nyquist, *Phys. Rev.* 32, 110 (1928)

HISTORY

In 1926, experimental physicist John Johnson working in the physics division at Bell Labs was researching noise in electronic circuits. He discovered that there was an irreducible low level of noise in resistors whose power was proportional to temperature. Harry Nyquist, a theorist in that division, got interested in the phenomenon and developed an elegant explanation based on fundamental physics.

THEORY OF THERMAL JOHNSON NOISE

Thermal agitation of electrons in a resistor gives rise to random fluctuations in the voltage across its terminals, known as Johnson noise. In Problem 1, you are to show that in a narrow band of frequencies, Δf, the contribution to the mean-squared noise voltage from this thermal agitation is,

$$\left\langle V(t)^2 \right\rangle_{time} = 4Rk_BT\Delta f \tag{1}$$

where R is the resistance in ohms and T is the temperature in degrees Kelvin for the resistor, k_B is the Boltzmann constant (1.38×10^{-23} J/K).

This voltage is usually too small to be detected without amplification. If the resistor is connected across the input of a high-gain amplifier whose voltage gain as a function of frequency is $G(f)$, the mean square of the voltage output of the amplifier will be:

$$\left\langle V(t)^2 \right\rangle_{time} = 4Rk_BT\int_0^\infty [G(f)]^2\, df + \left\langle V(t)_N^2 \right\rangle_{time} \tag{2}$$

where $\left\langle V(t)_N^2 \right\rangle_{time}$ is the output noise generated by the amplifier itself.

Thus by measuring $\langle V(t)^2 \rangle_{time}$ as a function of R and making a plot, one obtains $4k_B T \int_0^\infty [G(f)]^2 df$ from the slope, while the abscissa gives $\langle V(t)_N^2 \rangle_{time}$. But the amplifier gain $G(f)$ can be independently measured and the gain integral $\int_0^\infty [G(f)]^2 df$ evaluated. The slope will then give a value for the Boltzmann constant k_B.

This is in outline the first part of the experiment. The second part involves measuring the noise voltage as a function of the temperature, to verify the expected temperature dependence.

Problem 1 - Derivation of Eq. (1)

An electrical transmission line connected at one end to a resistor R and at the other end by an "equivalent" resistor R may be treated as a one- dimensional example of black body radiation.

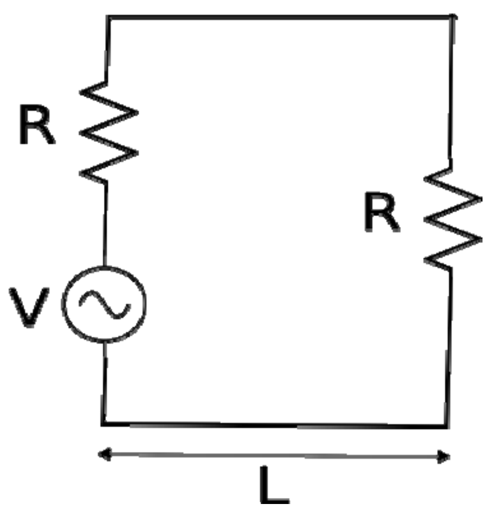


Figure 1. Two resistors (of equal resistance R) coupled via a transmission line.

At finite temperature T, the resistor R generates a noise voltage $V(t)$ which will propagate down the line. If the characteristic impedance of the transmission line is made equal to R, the radiation incident on the "equivalent" resistor R from the first resistor R should be completely absorbed.

The permitted standing wave modes in the line have $\lambda = 2L/n$ and $f = (c/2L)n$, where $n = 1, 2, 3,$ etc., and v is the wave velocity in the line. The separation of the modes in frequency is $v/2L$ and the number of modes between f and $f + \Delta f$ is

$$\sigma(f)\Delta f = (2L/c)\Delta f \qquad (3)$$

From the Planck distribution or the equipartition theorem, the mean thermal energy contained in each electromagnetic mode or photon state in the transmission line is:

$$\langle E(f)\rangle = \frac{hf}{e^{hf/k_BT}-1} \approx k_BT \quad \text{at low frequencies.} \tag{4}$$

From Eq. (3) and (4) find the electromagnetic energy $\langle E(f)\rangle \sigma(f)\Delta f$ in a frequency interval Δf . One half of this energy is generated by the first resistor of R and propagating towards the "equivalent" resistor R . Knowing the propagation time from the generating resistor to the absorbing resistor $\Delta t = L/c$, show that the absorbed power by the "equivalent" resistor R equals

$$P(f)\Delta f = k_BT\Delta f . \tag{5}$$

In thermal equilibrium, this power is simply the ohmic heating generated by a noise voltage source $V(t)$ from the first resistor. Since $V(t)$ is terminated by the absorbing resistor R and has an "internal" resistance R (the first resistor), it produces a current $I = V/(2R)$ in the line. Hence the power absorbed by the "equivalent" resistor R over the frequency interval Δf can also be calculated as

$$I^2R = \left(\frac{V}{2R}\right)^2 R = \frac{V^2}{4R} = \frac{V^2(f)\Delta f}{4R} \tag{6}$$

By equating $P(f)\Delta f = k_BT\Delta f$ to $V^2\Delta f / R$, show that

$$V^2(f)\Delta f = 4k_BTR\Delta f \tag{7}$$

and

$$\langle V(t)^2\rangle_{time} = 4k_BTR\Delta f \tag{8}$$

This is known as Nyquist's theorem as shown in Eq. (1). The power spectral density (noise power per unit frequency) is independent of frequency. Most other noise sources in nature have a f^{-1} to f^{-2} spectrum.

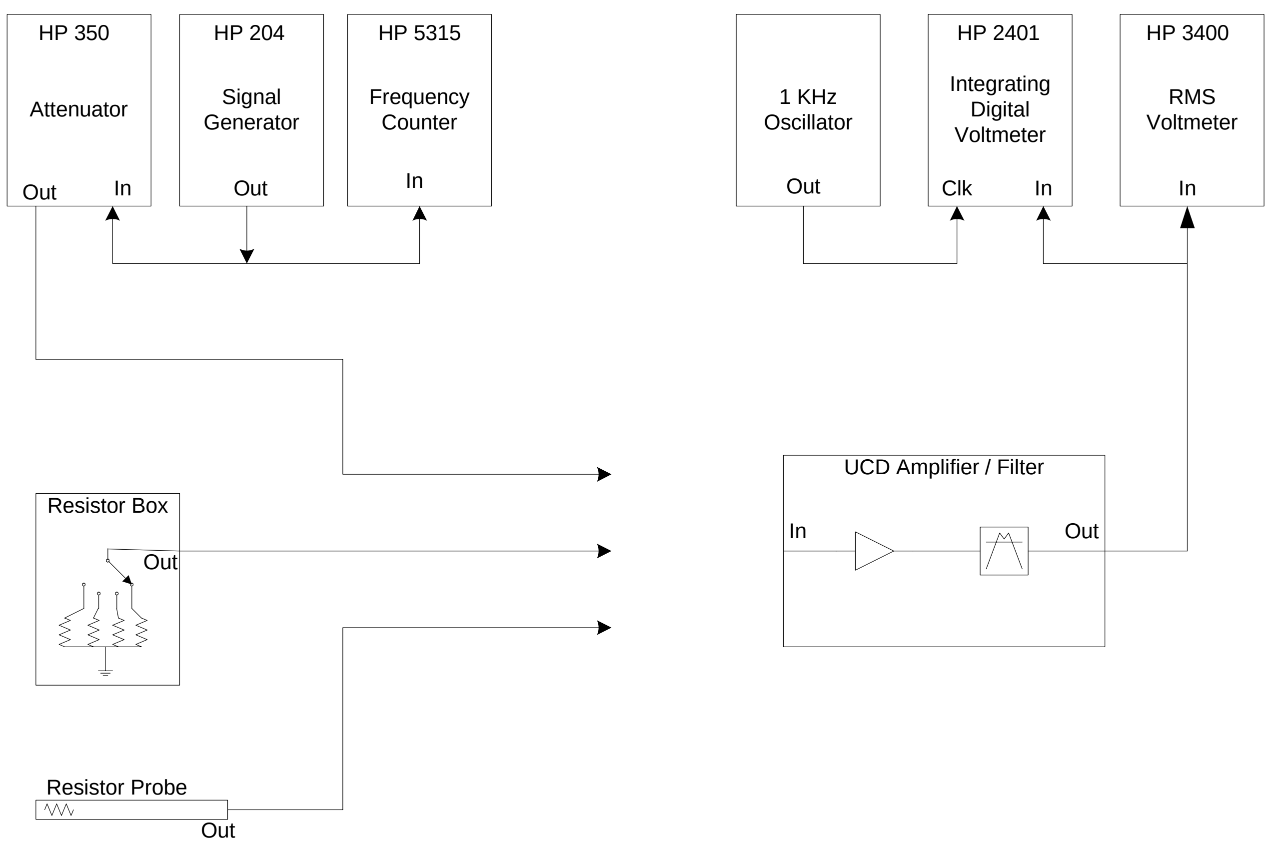
Question: what is the integrated power of this Johnson noise over all frequencies? [i.e., why can't a single resistor supply the world's energy needs?]

Eq. (1) is interesting: the left hand side describes random fluctuations of a lossy system in thermodynamic equilibrium. i.e. it is an equilibrium property; both the electrons and the lattice are in equilibrium at some temperature *T*. However, the right hand side of the equation refers explicitly to a non-equilibrium property of the same lossy system: the resistance *R*, which is measured by applying a voltage (taking the system out of equilibrium) and measuring the current of electrons scattering through the lossy system. **Johnson noise is an example of a broader fundamental principle in nature called the "*fluctuation-dissipation theorem*." It relates the non-equilibrium dissipation in a system to its spontaneous fluctuations in equilibrium. See accompanying papers by Callen.**

SAMPLE AND APPARATUS

A very low noise operational amplifier is used as the first stage of amplification for the Johnson noise. Roll-off filters limit the bandwidth on the low frequency side, while parasitic capacitance shunting the resistors will limit it on the high frequency side. As a result, the bandwidth is approximately 1 kHz. The apparatus is connected with shielded coaxial cables as shown to reduce pickup.

The sine wave oscillator is used to measure the gain of the amplifier. The oscillator output is put through an attenuator to reduce it to the level needed to be able to insert into the amplifier. The attenuation factor of the attenuator is accurately given by the controls and does not need to be calibrated. Its output should be directly connected to the amplifier EXT input to avoid voltage drops in connecting cables. The frequency *f* of the oscillator can be accurately set and determined with the Integrating Digital Voltmeter. Check with the Instructor or T.A. for how to set it up.



PROCEDURE FOR MEASURING THE GAIN INTEGRAL $\int_0^\infty [G(f)]^2 df$

You obtain $\int_0^\infty [G(f)]^2 df$ by measuring the amplifier gain G(f) at a discrete and evenly spaced set of frequency values (f) and then evaluating the discrete sum $\sum [G(f)]^2 \Delta f$ numerically.

Let the amplifier warm up for at least half an hour before starting this process. The amplifier is powered by batteries. If the unit has not been used recently, batteries need to be checked. To measure G(f), set the input switch to EXT, connect a precision broadband voltage attenuator to the input of the amplifier. The attenuator is used to assist you in determining the amplifier gain G(f) as follows.

Supply the input of the attenuator with a sinusoidal voltage signal of $V_i = 1$ volt at a frequency between zero and 4000 Hz. For the best accuracy, measure the applied voltage signal with either a digital voltmeter or an oscilloscope. Since at f = 1000 Hz, G(f) is roughly 10000, you can set the attenuation parameter $N_{dB} \sim 80$ dB that gives a voltage attenuation factor of $G_A = 10^{N_{dB}/20} \sim 10000$). The output of the attenuator, V_i/G_A, is then fed into the input of the amplifier. You measure the amplifier output $V_o = (V_i/G_A)G(f)$ with the same voltmeter or the oscilloscope. The amplifier gain is given by

$$G(f) = (V_o/V_i)G_A = (V_o/V_i)10^{N_{dB}/20} \tag{9}$$

It is best that you adjust N_{dB} so that $V_o/V_i \sim 1$. Repeat the measurement at a series of frequencies, to obtain a discrete set of G(f) values. From these results, you can numerically calculate $\int_0^\infty [G(f)]^2 df$.

JOHNSON NOISES FROM RESISTORS AT ROOM TEMPERATURE:

Disconnect the attenuator from the amplifier. Connect the resistor box to the amplifier and select a *R* value using the switch on the box. Use the HP3400A RMS Voltmeter to measure the rms voltage of the amplified Johnson noise signal. Since the rms voltage fluctuates on the time scale of a fraction of second, it is difficult to obtain an accurate reading of the mean of the rms voltage. To obtain the latter, you use an HP240IC Integrating Digital Voltmeter with the following procedure.

(1) Connect the DC output at the rear panel of the HP3400A RMS Voltmeter to "Hi" and "Lo" on the front panel of HP240IC Integrating Digital Voltmeter;
(2) On the front panel, set "Function" to "VOLT";
(3) Set "Range " to "10V";
(4) Set "SAMPLE PERIOD" to "1 SEC";
(5) Send a 1000-Hz sinusoidal signal from a HP20 Oscillator to "External Clock Input" at the rear of the Integrating Digital Voltmeter;
(6) Set the frequency standard (STD) next to "External Clock Input" to "EXT";
(7) Wait for 100 seconds before the voltage integration and average is complete. The displayed voltage value V_d is the 100-second average of the DC output multiplied by 100. To convert this value to the rms value of the amplified Johnson noise V_o, you need to divide V_d by 100 and then multiply the scale on the front panel of the HP3400A RMS Voltmeter.

Measure the noise voltage of each of the resistors. The value of the resistance of each of the resistors is written on the amplifier box. If you wish to check the resistance of these resistors, you may by connecting an ohm meter to the resistor box connector and measuring the selected *R*.

Use Eq. (2) to calculate the Boltzmann constant k_B, taking into account the corrections mentioned below. Also, you should compare the value of the amplifier noise, $\langle V_N^2 \rangle_{time}$, obtained from your data of the noise voltage measured at the amplifier output when the input is shorted.

TEMPERATURE DEPENDENCE OF THE JOHNSON NOISE

A shielded resistor in a sealed $\frac{1}{2}$-inch diameter stainless steel tube is provided to explore the temperature dependence of Johnson noise. The interior of the tube is filled with helium gas for thermal contact between resistor and the outside. Connect this probe directly to the INPUT connector on the amplifier (additional cable will only add capacitance and microphonic noise).

Record the RMS voltage produced by this resistor at room temperature (~ 300 K as measured with thermometer), and at liquid nitrogen (77 K) and liquid helium (4.2 K). For low temperature measurements, make sure that the probe is filled with helium gas before it is immersed in the containers of liquid nitrogen and liquid helium. The helium gas will not become liquefied, and will help cool the resistor to the final temperature by conducting the heat away from it.

Plot the rms noise voltage as a function of the temperature. Also, measure the resistance of the resistor at each of the temperatures (since the resistance of most resistors is a strong function of the temperature).

If you find any discrepancy between the measurement and the theory, suggest what their source(s) might be.

MEASURING NOISE SPECTRA WITH A LabVIEW VIRTUAL INSTRUMENT

In this section of the laboratory, you

(1) learn how to use a computer-aided data acquisition method (LabVIEW virtual instrument) to perform voltage measurement;

(2) measure thermal Johnson noise power spectra or $V^2(f)$ using a Fast-Fourier-Transform program on LabVIEW and verify that thermal Johnson noise is indeed frequency-independent;

(3) determine the resistance and temperature dependence of the noise spectra and in turn calculate the Boltzmann constant k_B;

(4) (Optional) use Johnson noise spectra to determine the frequency response of an amplifier gain G(f).

The LabVIEW program for the thermal Johnson noise is called "Johnson_Noise _2002.vi." It is in the Physics 122 folder on the PC computer by the experiment. It is placed in the "Physics122Lab_folder" on the C Drive. It is ready to be used to measure the noise spectra for the various fixed resistors in the amplifier box.

The program measures the noise voltage V(t) by digitizing a voltage input using an analog-to-digital converter on a data acquisition board inside the computer. The board measures a user-set total number of samples N_s with a user-set sampling rate f_r. f_r is equal to twice of the maximum measurable frequency f_{max}. This is because that one needs to sample at least two time points on a sine wave to determine its frequency. This is known as the Nyquist sampling theorem.

The program then computes the Fourier transform V(f) of the measured voltage V(t) by using a computer algorithm called the Fast Fourier Transform (FFT), invented by Cooley and Tukey. This algorithm is much more efficient if the total number of samples $N_s = 2^n$, with n being an integer. Using this algorithm, the maximum number of data points obtained over the frequency range from 0 to f_{max} is $N_f = N_s/2$. The LabVIEW panel displays the real-time signal and the power spectral density $V^2(f)$ of each run.

Note that for each measurement of N_s samples, the power spectral density (PSD) is quite noisy. We improve the signal-to-noise ratio of the measured power spectral density by averaging N such spectra. The signal-to-noise ratio is improved by $\sqrt{N}$. In the LabVIEW panel, the averaged power spectrum $V^2(f)\Delta f$ where $\Delta f = f_{max}/N_f = f_r/N_s$ is the frequency interval between the data points.

Note that the time-averaged mean squared total noise $\langle V^2(t) \rangle$ equals the mean squared total noise in frequency space, i.e.,

$$\langle V^2(t) \rangle = \int_0^\infty V^2(f)\Delta f = \sum_{n=1}^{N_f} V_n^2(f)\Delta f. \tag{10}$$

The program calculates the sum $\sum_{n=1}^{N_f} V_n^2(f)\Delta f$ that yields $\langle V^2(t) \rangle$.

Finally, measurements of noise are very important to physics experiments, because the actual noise levels in the experiment can determine whether one can measure small signal levels in the experiment. Measurements of noise power spectra as described here are frequently performed to understand the sources of the noise in the experiment. If you understand the noise in your experiment, you can then work to reduce noise sources by, for example, choosing components with less noise, averaging longer to reduce the effects of noise on the signal, or working in frequency regions where the noise is lower. In fact, specialized frequency analyzers exist; these are instruments which can easily measure such noise spectra, and they work the same way your LabVIEW computer program does. Some modern digital sampling oscilloscopes has a useful FFT option, and can be used to explore a wide frequency spectrum. Not only are there other sources of noise, there are also other sources of interference which may introduce systematic errors. For example, the local FM station has a particularly strong signal in the lab (you should look for this at ~103MHz and be sure it is not present at the low-level parts of your circuit by probing with a scope or spectrum analyzer. Even if your scope does not have a FFT option you can change the time base and sensitivity to see this sine wave if it is present.)

This noise power spectrum measurement by a computer and fast Fourier Transform is particularly useful for measuring the noise of the resistor in the separate probe as a function of temperature (room temperature (~300K), in liquid nitrogen (77K), and in liquid helium (4.2K). The noise from this resistor is particularly susceptible to microphonic noise. Microphonic noise is the noise voltage generated in electric wires due to their motion through capacitive effect or piezo-electric effect. Thus it can be generated from the probe being shaken, by people walking in the room causing vibrations in the probe, etc. Measuring the noise power spectrum allows you to distinguish the Johnson noise (which is not frequency dependent) from microphonic noise and line frequency noise that peak at specific frequencies such as multiples of 60Hz.

(1) **Learning a computer-aided data acquisition with a LabVIEW virtual instrument:**

Consult with the T.A. or an instructor of the Physics 122 Lab on how the LabVIEW works and quickly explain how "Johnson_Noise _2002.vi" operates. You should make an effort to familiarize yourself with the concept and strategy of a LabVIEW virtual instrument for computer-aided data acquisition.

For your experiment, you need to create your own folder to store the data files and any LabVIEW programs that you have created or saved in your own "name". You may want to bring a floppy diskette (you can buy them at the bookstore) and back up your files onto it, so that if anything should happen to the computer or its hard disk, your files will not be lost.

(2) **Measure thermal Johnson noise power spectra or** $V^2(f)\Delta f$ **using "Johnson noise power spectrum analyzer" and verify that thermal Johnson noise is indeed frequency-independent**

Question: How does the noise power spectrum $V^2(f)\Delta f$ compare to the square of the amplifier gain $G^2(f)$ that you measured ?

(3) **Determine the resistance and temperature dependence of the noise spectra and in turn calculate the Boltzmann constant** k_B

Question: How does the noise power spectrum $V^2(f)\Delta f$ (shape and magnitude) vary with the resistance R ?

Question: How does the noise power spectrum $V^2(f)\Delta f$ vary with temperature T ?

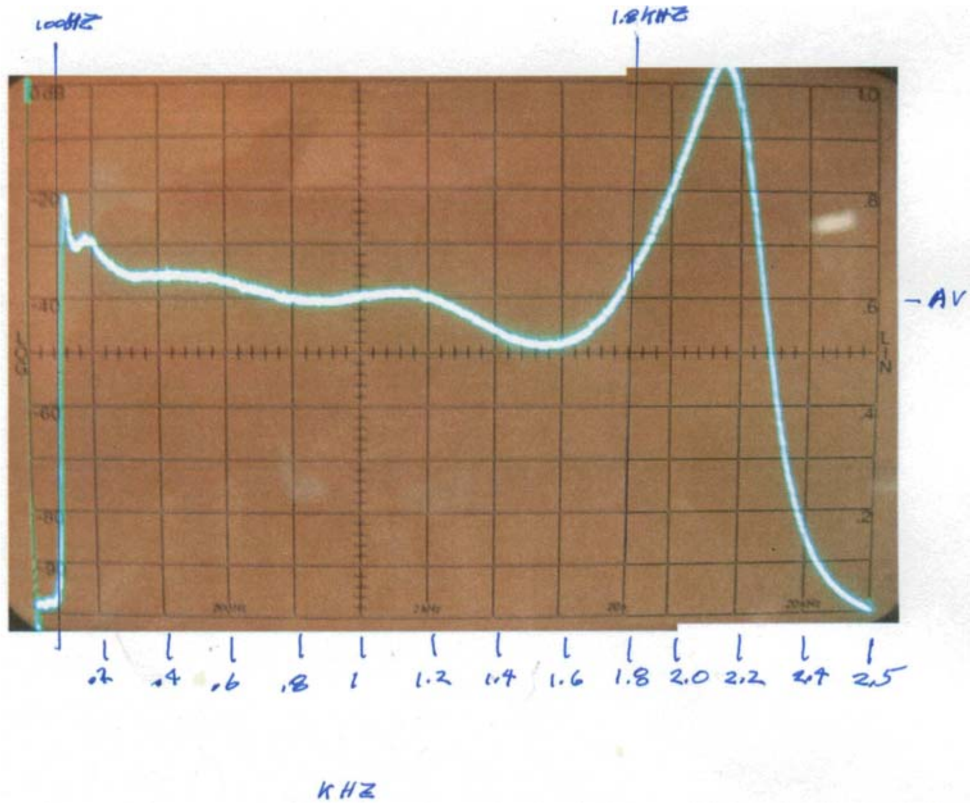
Question: Can you think of a way to use the noise power spectrum and the measured gain curve $G^2(f)$ to calculate the Boltzmann constant k_B without the AC Voltmeter and the Integrating Voltmeter ?

(4) (Optional) **Use Johnson noise spectra to determine the frequency response of an amplifier gain G(f)**

Since the thermal Johnson noise from a resistor is a broad frequency-band generator with a constant power spectral density $V^2(f) = 4k_B TR$, one can use the amplified thermal Johnson noise to measure the amplifier gain G(f) using a fast Fourier transform method with a LabVIEW program. Consult the instructor or T.A. for this option.

Apparatus

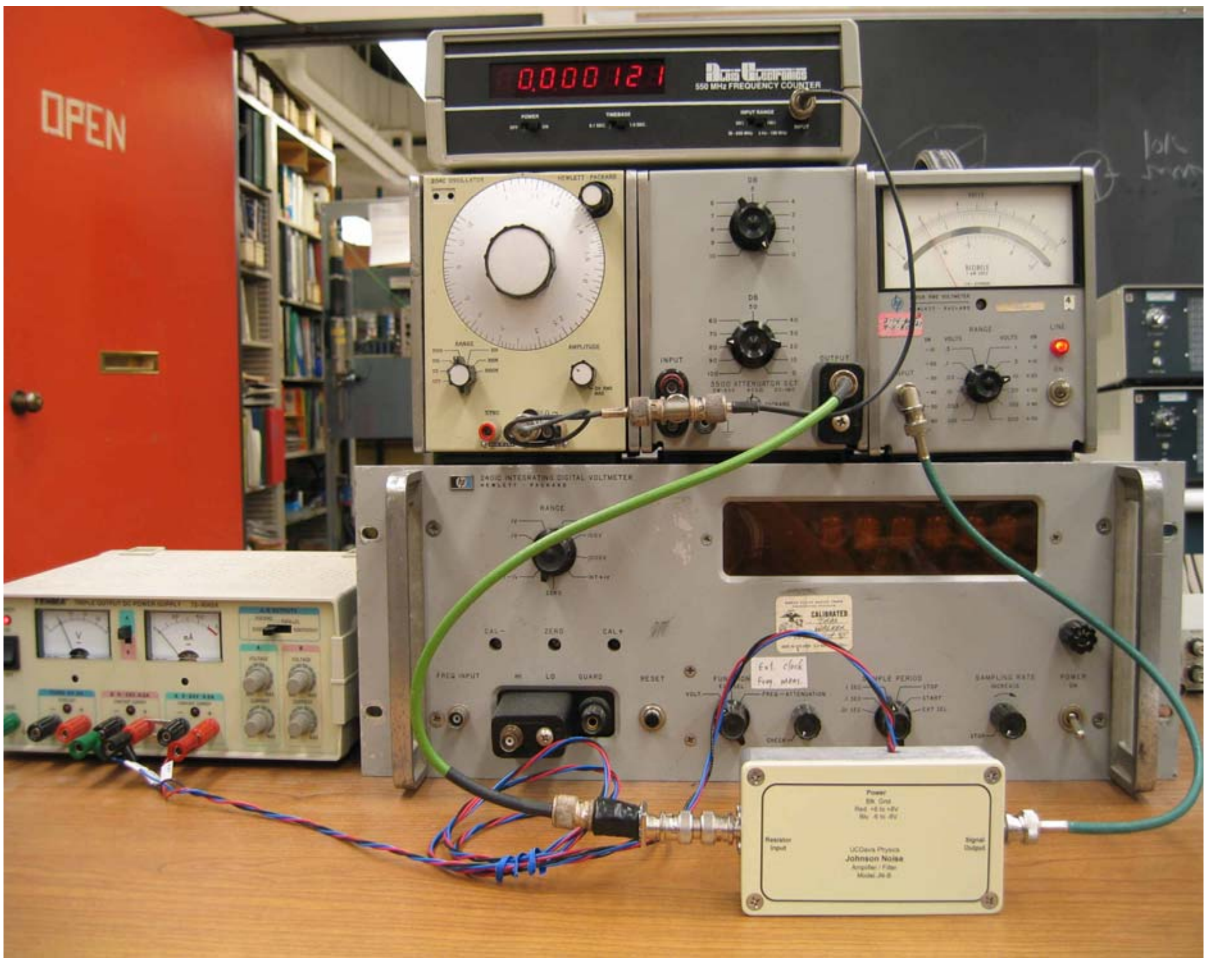
There are three units built specifically for this experiment. The first box is combination amplifier and filter. A very low noise operational amplifier is used as the first stage of amplification for the Johnson noise. Three total amplification stages are used for a total gain of about 10,000. Active filters are included with the amplifier to cut out the low and high frequencies, leaving a pass band covering approximately 100 Hz to 2 Khz. The box is a prototype and the filter characteristics are still a little strange. The photo below shows the filter characteristics on a network analyzer screen. The filter roll off is fairly sharp on both ends, but there is evident peaking at the edges of the pass band.



There is a resistor box that attaches to the amplifier input to allow testing for Johnson noise on several different values of resistors and a probe with a 1.2 Megohm resistor that is used for testing of temperature effects on Johnson noise. When testing resistors for Johnson noise, the signal generator, frequency counter and attenuator are not used. When checking for amplifier/filter characteristics, the Generator, counter and attenuator are connected in place of the resistor box or probe. Amplifier output is read on the HP 3400 Analog AC Volt Meter.

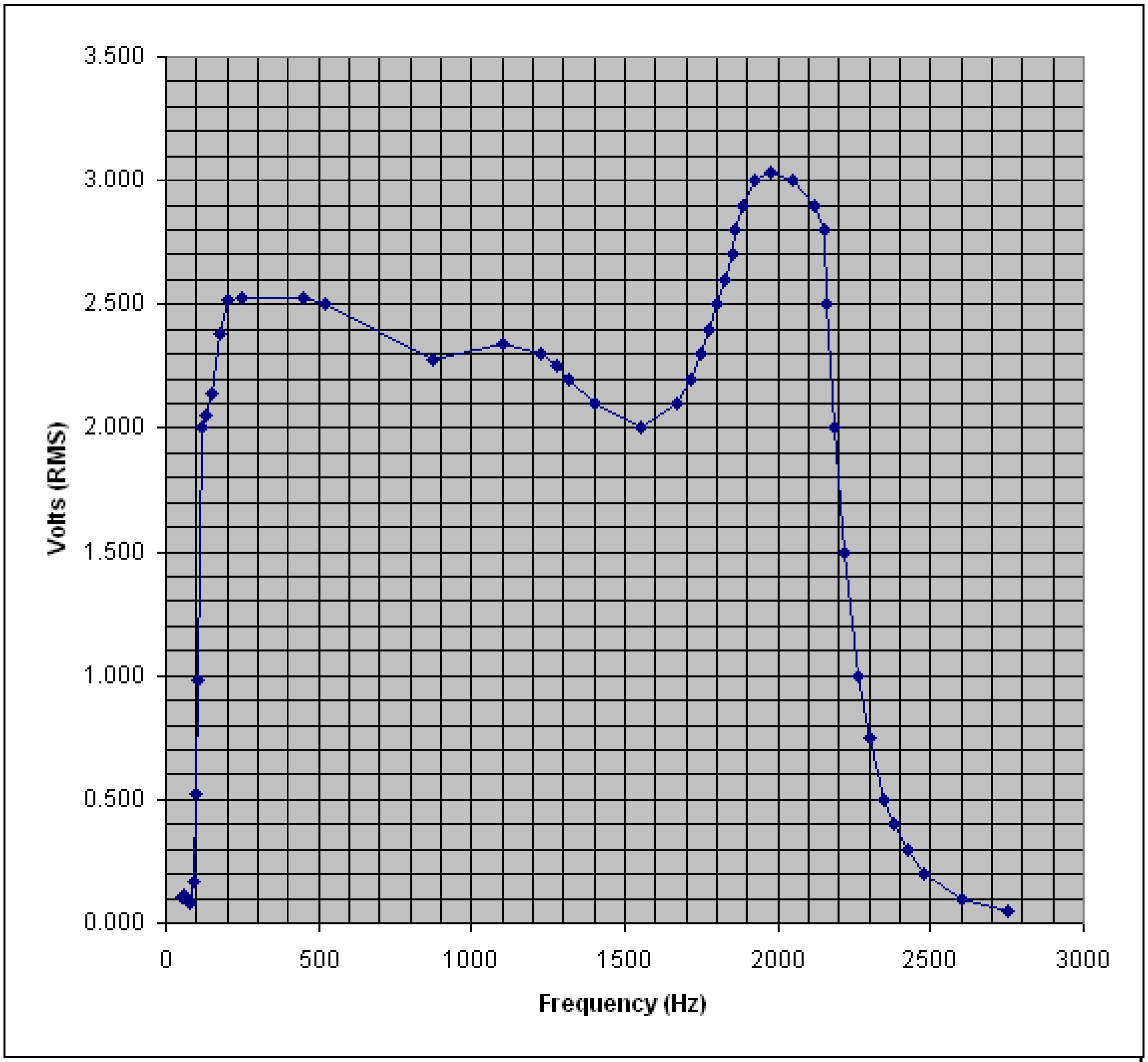
Measuring the Gain Integral

Before taking data for Johnson noise, the characteristics of the Amplifier and Filter must be accurately determined. The setup for this is shown in the photo below.



You can obtain the curve, or waveform, of the Amp/Filter by applying an input sine wave of known frequency and amplitude and then measuring the output voltage. Measuring the output at many frequencies across the band of interest and plotting the results on a frequency vs amplitude graph will provide results such as those on the next page.

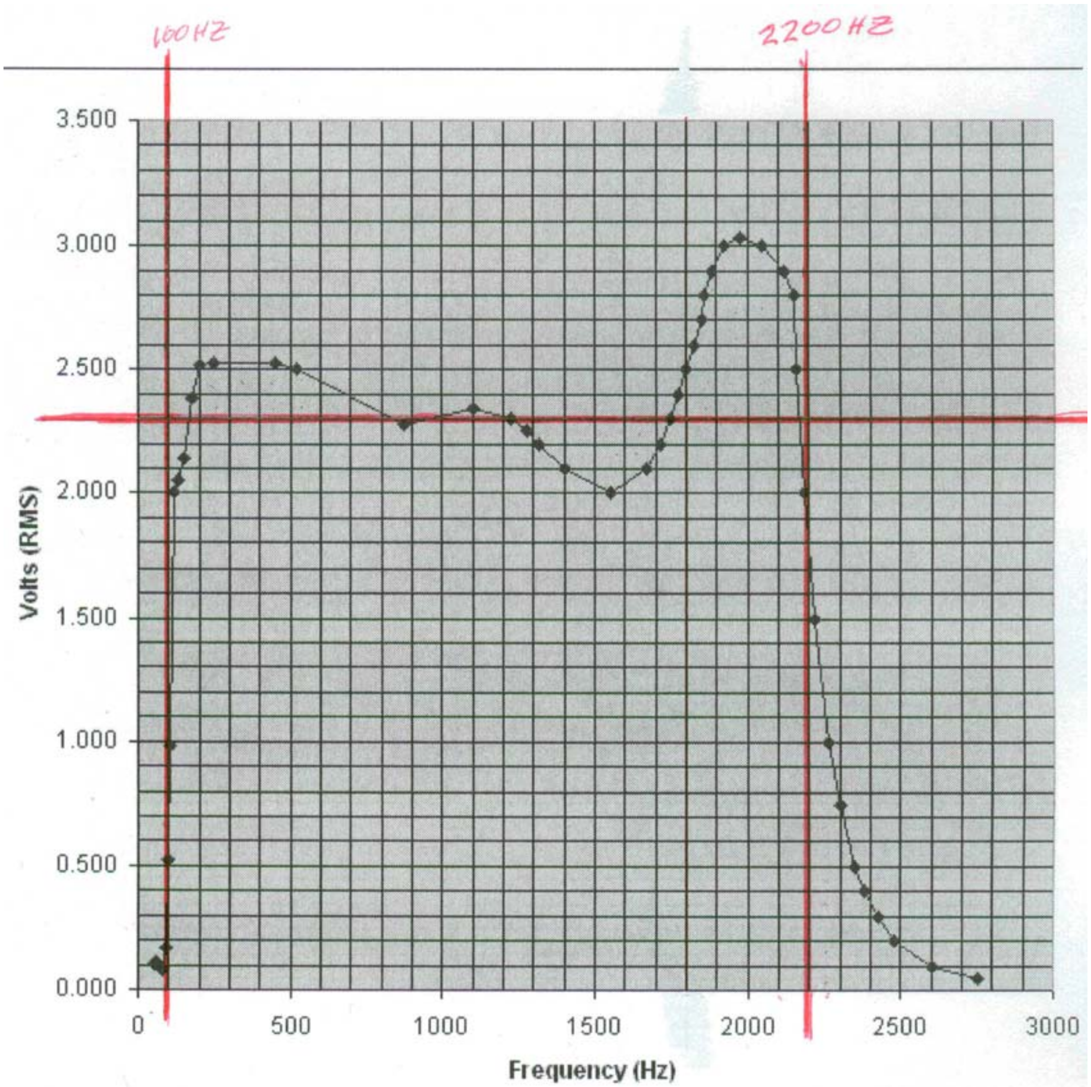
Volts	Hz
0.108	50
0.112	60
0.078	80
0.167	90
0.520	100
0.980	105
2.000	120
2.050	130
2.140	150
2.380	175
2.520	200
2.530	250
2.530	450
2.500	520
2.280	875
2.340	1100
2.300	1225
2.250	1280
2.200	1320
2.100	1400
2.000	1550
2.100	1667
2.200	1715
2.300	1750
2.400	1775
2.500	1800
2.600	1825
2.700	1850
2.800	1860
2.900	1885
3.000	1925
3.030	1975
3.000	2050
2.900	2120
2.800	2150
2.500	2160
2.000	2185
1.500	2220
1.000	2265
0.750	2300
0.500	2350
0.400	2380
0.300	2425
0.200	2480
0.100	2600
0.050	2750



If the filter response plot above simply had nice vertical sides and a flat top, then the Johnson noise over the bandwidth would be the power spectral density times the bandwidth. Any filter response may be represented by an *effective* bandwidth determined by the total area under the filter response curve.

The full bandwidth internal noise can be measured by connecting a short circuit adapter (shunt) across the amplifier input and measuring the output voltage.

Since our filter response does not look like a nice rectangular box we will have to integrate the curve. We can then calculate a correction factor to adjust our measured readings. A copy of the plot marked with a handy box to use as our bandwidth is shown below. The top line is the amplifier output before filtering. The experiment does not allow access to this point right now, but bench testing indicates it should be in the position shown.



The area under this curve is $V_{in}\int_0^\infty [G(f)]^2 df$, where V_{in} is the applied sine wave amplitude. The effective bandwidth is simply the width Δf_{eff} of a rectangular passband which has total area $\int_0^\infty [G(f)]^2 df$.

i.e. $\Delta f_{eff} = \int_0^\infty [G(f)]^2 df \, / [G(f_0)]^2$ where f_0 is taken at the center of the passband.