Solution to 9C-A (2010) Midterm #2

1- (a)

$C_2$ and $C_3$ are in series, and can be represented by an equivalent capacitor $C_{23}$

$$C_{23} = \frac{1}{\frac{1}{C_2} + \frac{1}{C_3}} = \frac{C_2 C_3}{C_2 + C_3} = 4 \mu F$$

$C_1$ and $C_{23}$ are in parallel, and can be represented by an equivalent capacitor $C_{123}$

$$C_{123} = C_1 + C_{23} = 12 \mu F$$

$C_4$ and $C_{123}$ are in series, and can be represented by an equivalent capacitor $C_{1234}$ (the one we are looking for)

$$C_{1234} = \frac{1}{\frac{1}{C_4} + \frac{1}{C_{123}}} = \frac{C_4 \cdot C_{123}}{C_4 + C_{123}} = 4.8 \mu F$$

1- (b)

The charge on $C_4$ is the same as the charge on the $C_{1234}$, thus

$$Q_4 = C_{1234} \cdot V_{ab} = 96 \mu C$$

The potential difference $V_{ab} = V_a - V_b$ equals to the charge on $C_{1234}$ divided by $C_{123}$ as $C_{123}$ is in series with $C_4$, ...
\[ V_{ad} = \frac{Q_4}{C_{123}} = \frac{96 \mu C}{12 \mu F} = 8 V \]

Thus the charge on \( C_1 \) is given by
\[ Q_1 = C_1 \cdot V_{ad} = 64 \mu C \]

The charges on \( C_2 \) and \( C_3 \) are the same, and given by
\[ Q_2 = Q_3 = C_{23} \cdot V_{ad} = 32 \mu C \]

\(-c\) The potential difference across \( C_4 \) is
\[ V_4 = \frac{Q_4}{C_4} = \frac{96 \mu C}{8 \mu F} = 12 V \]

The potential difference across \( C_1 \) is
\[ V_1 = \frac{Q_1}{C_1} = 8 V \ (= V_{ad} - V_4) \]

The potential differences across \( C_2 \) and \( C_3 \) are the same, and equal to one half of \( V_1 \):
\[ V_2 = V_3 = \frac{V_1}{2} = 4 V \]

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2-(a) With the dielectric slab inserted, the capacitor can be treated as 3 capacitors connected in the following way.

\[ C_L = \frac{\varepsilon_0 (2A/3)}{d-a} \left( \frac{\varepsilon_0 K (2A/3)}{a} \right) \]

\[ = \frac{\varepsilon_0 (2A/3)}{d} \cdot \frac{K \cdot d}{a + K (d-a)} \]

The equivalent capacitance of the right side is

\[ C_R = \frac{\varepsilon_0 (A/3)}{d} \]
As a result,

\[ C = C_L + C_R \]

\[ = \frac{e_0 A}{d} \cdot \frac{(2/3) k d}{a + k(d-a)} + \frac{e_0 A}{d} \frac{1}{3} \]

\[ = C_0 \left[ \frac{(2/3) k d}{a + k(d-a)} + \frac{1}{3} \right] \]

\[ = C_0 \left[ \frac{4d}{a + 6(d-a)} + \frac{1}{3} \right] \]

2-(b) When \( a = 0 \),

\[ C = C_0 \]

2-(c) When \( a = d \),

\[ C = C_0 \left( \frac{4d}{d} + \frac{1}{3} \right) = \left( \frac{4}{3} \right) C_0. \]
3-(a) The equivalent resistance of $R_2$, $R_3$, and $R_4$ in parallel is

$$R_{234} = \frac{1}{\frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_4}} = \frac{R_1}{3} = 2\Omega.$$ 

The total resistance viewed from the two terminals of the circuit is

$$R_{1234} = R_1 + R_{234} = 8\Omega \quad \text{(in series)}$$

The current through $R_1$ and $R_{234}$ is the

$$I_1 = I_{234} = \frac{E}{R_{234}} = \frac{72V}{8\Omega} = 9\text{A}$$

The power dissipated in $R_1$ is

$$P_1 = I_1^2 \cdot R_1 = (9\text{A})^2 \cdot 6\Omega = 486\text{ Watts}.$$ 

The current through $R_2$ is $\frac{1}{3}$ of $I_{234} = 9\text{A}$, thus

$$P_2 = I_2^2 \cdot R_2 = \left(\frac{I_{234}}{3}\right)^2 \cdot R_2 = (3\text{A})^2 \cdot 6\Omega = 54\text{ Watts}.$$
3-(b) When \( R_4 \) is removed,
\[
R_{23} = \frac{1}{\frac{1}{R_2} + \frac{1}{R_3}} = 3\Omega.
\]

Thus the total resistance seen by the source is
\[
R_{123} = R_1 + R_{23} = 9\Omega.
\]

The current through \( R_1 \) and \( R_{23} \) is
\[
I_1 = I_{23} = I_{123} = \frac{E}{R_{123}} = \frac{72V}{9\Omega} = 8A.
\]

\[
P_1 = I_1^2R_1 = (8A)^2 \cdot 6\Omega = 384 \text{ Watt}
\]

\[
P_2 = P_3 = (I_{23/2})^2R_2 = (4A)^2 \cdot 2\Omega = 96 \text{ Watt}.
\]
4-(a) Assign the current in the clockwise direction.

Along the clockwise loop, starting from a, the potential drops add up to zero,

\[ 6I + 0.5I + 4 + 9I - 16 + 0.5I + 8I = 0 \]

\[ \therefore 24I = 12 \text{V} \quad I = +0.5 \text{A}. \]

So the current I flows in the direction as assigned and has a magnitude of 0.5 A.

4-(b) The terminal voltage across the 4-V battery is

\[ V_b - V_c = (V_b - V_f) + (V_f - V_c) = 0.5(0.5) + 4 \]

\[ = 4.25 \text{V} \text{rms} \]
4. (c) \[ V_{ad} = V_a - V_d = (V_a - V_e) + (V_e - V_s) + (V_s - V_d) \]

\[ = -0.5A(8\Omega) - 0.5A(0.5\Omega) + 16V \]

\[ = 1.75V \]
5- (a) Assign two currents $I_2$ and $I_1$ in the direct-current circuit as follows:

Along loop #1, start at point c, counterclockwise:

10 \left(I_2 + I_1\right) + 4I_1 - 5 + I_4 = 0 \quad \cdots (1)

3I_1 + 2I_2 = 1 \quad \cdots (1')

Along loop #2, starting at point a, counterclockwise:

\[-10 + 2I_2 - I_1 + 5 - 4I_1 + 3I_2 = 0 \quad \cdots (2)\]

\[-I_1 + I_2 = 1 \quad \cdots (2')\]

\[I_2 = I_1 + I_4 \quad \cdots (2'')\]

Insert (2'') into (1'):

\[5I_1 = -1 \quad \therefore I_1 = -\frac{1}{5} \text{ A} \quad \cdots (3)\]
\[ I_2 = I_1 + 1A = + \frac{4}{5}A \]
\[ (I_1 + I_2) \text{ passing } 10 \Omega = + (\frac{3}{5})A. \]

So, the current through 2Ω resistor branch is in the direction as assigned, and has a magnitude of 0.8A;

The current through 1Ω resistor branch is in the opposite direction to the assigned, and has a magnitude of 0.2A;

The current through 10Ω resistor branch is in the direction as assigned, and has a magnitude of 0.6A.

5- (b) \[ V_{ab} = V_a - V_b = (V_a - V_e) + (V_e - V_b) \]
\[ = - (3\Omega) I_2 + (4\Omega) I_1 \]
\[ = - (3\Omega)(0.8A) + (4\Omega)(-0.2A) \]
\[ = - 3.2 \text{ V} \]