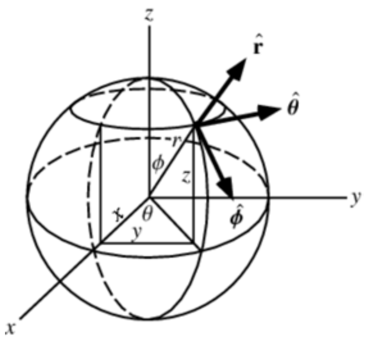
Spherical Coordinates



Spherical coordinates, also called spherical polar coordinates (Walton 1967, Arfken 1985), are a system of curvilinear coordinates that are natural for describing positions on a sphere or spheroid. Define θ to be the azimuthal angle in the xy-plane from the x-axis with $0 \leq \theta < 2\pi$ (denoted λ when referred to as the longitude), ϕ to be the polar angle (also known as the zenith angle and colatitude, with $\phi = 90° - \delta$ where δ is the latitude) from the positive z-axis with $0 \leq \phi \leq \pi$, and r to be distance (radius) from a point to the origin. This is the convention commonly used in mathematics.

In this work, following the mathematics convention, the symbols for the radial, azimuth, and zenith angle coordinates are taken as r, θ, and ϕ, respectively. Note that this definition provides a logical extension of the usual polar coordinates notation, with θ remaining the angle in the xy-plane and ϕ becoming the angle out of that plane. The sole exception to this convention in this work is in spherical harmonics, where the convention used in the physics literature is retained (resulting, it is hoped, in a bit less confusion than a foolish rigorous consistency might engender).

Unfortunately, the convention in which the symbols θ and ϕ are reversed is also frequently used, especially in physics. The symbol ρ is sometimes also used in place of r, and φ and ψ instead of ϕ. The following table summarizes a number of conventions used by various authors; be very careful when consulting the literature.

(radial, azimuthal, polar)	reference
(r, θ, ϕ)	this work, Zwillinger (1985, pp. 297-298)
(ρ, θ, ϕ)	Beyer (1987, p. 212)
(r, ϑ, φ)	Korn and Korn (1968, p. 60)
(r, ϕ, θ)	Misner et al. (1973, p. 205)
(`Rr`, `Pphi`, `Ttheta`)	`SetCoordinates[Spherical[`r, Ttheta, Pphi]] in the *Mathematica* package `VectorAnalysis``)
(r, φ, θ)	Arfken (1985, p. 102)
(r, ψ, θ)	Moon and Spencer (1988, p. 24)

The spherical coordinates (r, θ, ϕ) are related to the Cartesian coordinates (x, y, z) by

$$r = \sqrt{x^2 + y^2 + z^2} \quad (1)$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) \quad (2)$$

$$\phi = \cos^{-1}\left(\frac{z}{r}\right), \quad (3)$$

where $r \in [0, \infty)$, $\theta \in [0, 2\pi)$, and $\phi \in [0, \pi]$, and the inverse tangent must be suitably defined to take the correct quadrant of (x, y) into account.

In terms of Cartesian coordinates,

$$x = r\cos\theta\sin\phi \quad (4)$$

$$y = r\sin\theta\sin\phi \quad (5)$$

$$z = r\cos\phi. \quad (6)$$

The scale factors are

$$h_r = 1 \quad (7)$$

$$h_\theta = r\sin\phi \quad (8)$$

$$h_\phi = r, \quad (9)$$

so the metric coefficients are

$$g_{rr} = 1 \quad (10)$$

$$g_{\theta\theta} = r^2\sin^2\phi \quad (11)$$

$$g_{\phi\phi} = r^2. \quad (12)$$

The line element is

$$d\mathbf{s} = dr\,\hat{\mathbf{r}} + r\,d\phi\,\hat{\boldsymbol{\phi}} + r\sin\phi\,d\theta\,\hat{\boldsymbol{\theta}}, \quad (13)$$

the area element

$$d\mathbf{a} = r^2 \sin\phi \, d\theta \, d\phi \, \hat{\mathbf{r}}, \tag{14}$$

and the volume element

$$dV = r^2 \sin\phi \, d\phi \, d\theta \, dr. \tag{15}$$

The Jacobian is

$$\left| \frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} \right| = r^2 \sin\phi. \tag{16}$$

The position vector is

$$\mathbf{r} \equiv \begin{bmatrix} r\cos\theta\sin\phi \\ r\sin\theta\sin\phi \\ r\cos\phi \end{bmatrix}, \tag{17}$$

so the unit vectors are

$$\hat{\mathbf{r}} \equiv \frac{\frac{d\mathbf{r}}{dr}}{\left|\frac{d\mathbf{r}}{dr}\right|} \tag{18}$$

$$= \begin{bmatrix} \cos\theta\sin\phi \\ \sin\theta\sin\phi \\ \cos\phi \end{bmatrix} \tag{19}$$

$$\hat{\boldsymbol{\theta}} \equiv \frac{\frac{d\mathbf{r}}{d\theta}}{\left|\frac{d\mathbf{r}}{d\theta}\right|} \tag{20}$$

$$= \begin{bmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{bmatrix} \tag{21}$$

$$\hat{\boldsymbol{\phi}} \equiv \frac{\frac{d\mathbf{r}}{d\phi}}{\left|\frac{d\mathbf{r}}{d\phi}\right|} \tag{22}$$

$$= \begin{bmatrix} \cos\theta\cos\phi \\ \sin\theta\cos\phi \\ -\sin\phi \end{bmatrix}. \tag{23}$$

Derivatives of the unit vectors are

$$\frac{\partial\hat{\mathbf{r}}}{\partial r} = \mathbf{0} \tag{24}$$

$$\frac{\partial\hat{\boldsymbol{\theta}}}{\partial r} = \mathbf{0} \tag{25}$$

$$\frac{\partial\hat{\boldsymbol{\phi}}}{\partial r} = \mathbf{0} \tag{26}$$

$$\frac{\partial\hat{\mathbf{r}}}{\partial\theta} = \sin\phi\,\hat{\boldsymbol{\theta}} \tag{27}$$

$$\frac{\partial\hat{\boldsymbol{\theta}}}{\partial\theta} = -\cos\phi\,\hat{\boldsymbol{\phi}} - \sin\phi\,\hat{\mathbf{r}} \tag{28}$$

$$\frac{\partial\hat{\boldsymbol{\phi}}}{\partial\theta} = \cos\phi\,\hat{\boldsymbol{\theta}} \tag{29}$$

$$\frac{\partial\hat{\mathbf{r}}}{\partial\phi} = \hat{\boldsymbol{\phi}} \tag{30}$$

$$\frac{\partial\hat{\boldsymbol{\theta}}}{\partial\phi} = \mathbf{0} \tag{31}$$

$$\frac{\partial\hat{\boldsymbol{\phi}}}{\partial\phi} = -\hat{\mathbf{r}}. \tag{32}$$

The gradient is

$$\nabla = \hat{\mathbf{r}}\frac{\partial}{\partial r} + \frac{1}{r}\hat{\boldsymbol{\phi}}\frac{\partial}{\partial\phi} + \frac{1}{r\sin\phi}\hat{\boldsymbol{\theta}}\frac{\partial}{\partial\theta}, \tag{33}$$

and its components are

$$\nabla_r\hat{\mathbf{r}} = \mathbf{0} \tag{34}$$

$$\nabla_\theta\hat{\mathbf{r}} = \frac{1}{r}\hat{\boldsymbol{\theta}} \tag{35}$$

$$\nabla_\phi\hat{\mathbf{r}} = \frac{1}{r}\hat{\boldsymbol{\phi}} \tag{36}$$

$$\nabla_r\hat{\boldsymbol{\theta}} = \mathbf{0} \tag{37}$$

$$\nabla_\theta\hat{\boldsymbol{\theta}} = -\frac{\cot\phi}{r}\hat{\boldsymbol{\phi}} - \frac{1}{r}\hat{\mathbf{r}} \tag{38}$$

$$\nabla_\phi\hat{\boldsymbol{\theta}} = \mathbf{0} \tag{39}$$

$$\nabla_r\hat{\boldsymbol{\phi}} = \mathbf{0} \tag{40}$$

$$\nabla_\theta\hat{\boldsymbol{\phi}} = \frac{1}{r}\cot\phi\,\hat{\boldsymbol{\theta}} \tag{41}$$

$$\nabla_\phi\hat{\boldsymbol{\phi}} = -\frac{1}{r}\hat{\mathbf{r}} \tag{42}$$

(Misner *et al*. 1973, p. 213, who however use the notation convention (r, ϕ, θ)).

The Christoffel symbols of the second kind in the definition of Misner *et al*. (1973, p. 209) are given by

$$\Gamma^r = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\frac{1}{r} & 0 \\ 0 & 0 & -\frac{1}{r} \end{bmatrix} \quad (43)$$

$$\Gamma^\theta = \begin{bmatrix} 0 & \frac{1}{r} & 0 \\ 0 & 0 & 0 \\ 0 & \frac{\cot\phi}{r} & 0 \end{bmatrix} \quad (44)$$

$$\Gamma^\phi = \begin{bmatrix} 0 & 0 & \frac{1}{r} \\ 0 & -\frac{\cot\phi}{r} & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (45)$$

(Misner *et al*. 1973, p. 213, who however use the notation convention (r, ϕ, θ)). The Christoffel symbols of the second kind in the definition of Arfken (1985) are given by

$$\Gamma^r = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -r\sin^2\phi & 0 \\ 0 & 0 & -r \end{bmatrix} \quad (46)$$

$$\Gamma^\theta = \begin{bmatrix} 0 & \frac{1}{r} & 0 \\ \frac{1}{r} & 0 & \cot\phi \\ 0 & \cot\phi & 0 \end{bmatrix} \quad (47)$$

$$\Gamma^\phi = \begin{bmatrix} 0 & 0 & \frac{1}{r} \\ 0 & -\sin\phi\cos\phi & 0 \\ \frac{1}{r} & 0 & 0 \end{bmatrix} \quad (48)$$

(Walton 1967; Moon and Spencer 1988, p. 25a; both of whom however use the notation convention (r, ϕ, θ)).

The divergence is

$$\nabla\cdot\mathbf{F} = \frac{\partial}{\partial r}A^r + \frac{2}{r}A^r + \frac{1}{r\sin\phi}\frac{\partial}{\partial\theta}A^\theta + \frac{1}{r}\frac{\partial}{\partial\phi}A^\phi + \frac{\cot\phi}{r}A^\phi, \quad (49)$$

or, in vector notation,

$$\nabla\cdot\mathbf{F} = \left(\frac{2}{r} + \frac{\partial}{\partial r}\right)F_r + \left(\frac{1}{r}\frac{\partial}{\partial\phi} + \frac{\cot\phi}{r}\right)F_\phi + \frac{1}{r\sin\phi}\frac{\partial F_\theta}{\partial\theta} \quad (50)$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 F_r) + \frac{1}{r\sin\phi}\frac{\partial}{\partial\phi}(\sin\phi F_\phi) + \frac{1}{r\sin\phi}\frac{\partial F_\theta}{\partial\theta}. \quad (51)$$

The covariant derivatives are given by

$$A_{j;k} = \frac{1}{g_{kk}}\frac{\partial A_j}{\partial x_k} - \Gamma^i_{jk}A_i, \quad (52)$$

so

$$A_{r;r} = \frac{\partial A_r}{\partial r} \quad (53)$$

$$A_{r;\theta} = \frac{1}{r\sin\phi}\frac{\partial A_r}{\partial\phi} - \frac{A_\theta}{r} \quad (54)$$

$$A_{r;\phi} = \frac{1}{r}\left(\frac{\partial A_r}{\partial\phi} - A_\phi\right) \quad (55)$$

$$A_{\theta;r} = \frac{\partial A_\theta}{\partial r} \quad (56)$$

$$A_{\theta;\theta} = \frac{1}{r\sin\phi}\frac{\partial A_\theta}{\partial\theta} + \frac{\cot\phi}{r}A_\phi + \frac{A_r}{r} \quad (57)$$

$$A_{\theta;\phi} = \frac{1}{r}\frac{\partial A_\theta}{\partial r} - \Gamma^i_{\phi r}A_i\frac{\partial A_\theta}{\partial\phi} \quad (58)$$

$$A_{\phi;r} = \frac{\partial A_\phi}{\partial r} - \Gamma^i_{\phi r}A_i = \frac{\partial A_\phi}{r} \quad (59)$$

$$A_{\phi;\theta} = \frac{1}{r\sin\phi}\frac{\partial A_\phi}{\partial\theta} - \frac{\cot\phi}{r}A_\theta \quad (60)$$

$$A_{\phi;\phi} = \frac{1}{r}\frac{\partial A_\phi}{\partial\phi} + \frac{A_r}{r}. \quad (61)$$

The commutation coefficients are given by

$$c^\mu_{\alpha\beta}\vec{e}_\mu = [\vec{e}_\alpha, \vec{e}_\beta] = \nabla_\alpha\vec{e}_\beta - \nabla_\beta\vec{e}_\alpha \quad (62)$$

$$[\hat{\mathbf{r}}, \hat{\mathbf{r}}] = [\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\theta}}] = [\hat{\boldsymbol{\phi}}, \hat{\boldsymbol{\phi}}] = \mathbf{0}, \quad (63)$$

so $c^\alpha_{rr} = c^\alpha_{\theta\theta} = c^\alpha_{\phi\phi} = 0$, where $\alpha = r, \theta, \phi$.

$$[\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}] = -[\hat{\boldsymbol{\theta}}, \hat{\mathbf{r}}] = \nabla_r \hat{\boldsymbol{\theta}} - \nabla_\theta \hat{\mathbf{r}} = \mathbf{0} - \frac{1}{r}\hat{\boldsymbol{\theta}} = -\frac{1}{r}\hat{\boldsymbol{\theta}}, \tag{64}$$

so $c_{r\theta}^{\theta} = -c_{\theta r}^{\theta} = -\frac{1}{r}$, $c_{r\theta}^{r} = c_{r\theta}^{\phi} = 0$.

$$[\hat{\mathbf{r}}, \hat{\boldsymbol{\phi}}] = -[\hat{\boldsymbol{\phi}}, \hat{\mathbf{r}}] = \mathbf{0} - \frac{1}{r}\hat{\boldsymbol{\phi}} = -\frac{1}{r}\hat{\boldsymbol{\phi}}, \tag{65}$$

so $c_{r\phi}^{\phi} = -c_{\phi r}^{\phi} = \frac{1}{r}$.

$$[\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}] = -[\hat{\boldsymbol{\phi}}, \hat{\boldsymbol{\theta}}] = \frac{1}{r}\cot\phi\,\hat{\boldsymbol{\theta}} - \mathbf{0} = \frac{1}{r}\cot\phi\,\hat{\boldsymbol{\theta}}, \tag{66}$$

so

$$c_{\theta\phi}^{\theta} = -c_{\phi\theta}^{\theta} = \frac{1}{r}\cot\phi. \tag{67}$$

Summarizing,

$$c^r = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{68}$$

$$c^\theta = \begin{bmatrix} 0 & -\frac{1}{r} & 0 \\ \frac{1}{r} & 0 & \frac{1}{r}\cot\phi \\ 0 & -\frac{1}{r}\cot\phi & 0 \end{bmatrix} \tag{69}$$

$$c^\phi = \begin{bmatrix} 0 & 0 & -\frac{1}{r} \\ 0 & 0 & 0 \\ \frac{1}{r} & 0 & 0 \end{bmatrix}. \tag{70}$$

Time derivatives of the position vector are

$$\dot{\mathbf{r}} = \begin{bmatrix} \cos\theta\sin\phi\,\dot{r} - r\sin\theta\sin\phi\,\dot{\theta} + r\cos\theta\cos\phi\,\dot{\phi} \\ \sin\theta\sin\phi\,\dot{r} + r\cos\theta\sin\phi\,\dot{\theta} + r\sin\theta\cos\phi\,\dot{\phi} \\ \cos\phi\,\dot{r} - r\sin\phi\,\dot{\phi} \end{bmatrix} \tag{71}$$

$$= \begin{bmatrix} \cos\theta\sin\phi \\ \sin\theta\sin\phi \\ \cos\phi \end{bmatrix}\dot{r} + r\sin\phi \begin{bmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{bmatrix}\dot{\theta} + r\begin{bmatrix} \cos\theta\cos\phi \\ \sin\theta\cos\phi \\ -\sin\phi \end{bmatrix}\dot{\phi} \tag{72}$$

$$= \dot{r}\,\hat{\mathbf{r}} + r\sin\phi\,\dot{\theta}\,\hat{\boldsymbol{\theta}} + r\,\dot{\phi}\,\hat{\boldsymbol{\phi}}. \tag{73}$$

The speed is therefore given by

$$v \equiv |\dot{\mathbf{r}}| = \sqrt{\dot{r}^2 + r^2\sin^2\phi\,\dot{\theta}^2 + r^2\dot{\phi}^2}. \tag{74}$$

The acceleration is

$$\ddot{x} = \begin{array}{l}(-\sin\theta\sin\phi\,\dot{\theta}\,\dot{r} + \cos\theta\cos\phi\,\dot{r}\,\dot{\phi} + \cos\theta\sin\phi\,\ddot{r}) - \\ \left(\sin\theta\sin\phi\,\dot{r}\,\dot{\theta} + r\cos\theta\sin\phi\,\dot{\theta}^2 + r\sin\theta\cos\phi\,\dot{\theta}\,\dot{\phi} + r\sin\theta\sin\phi\,\ddot{\theta}\right) + \left(\cos\theta\cos\phi\,\dot{r}\,\dot{\phi} - r\sin\theta\cos\,\dot{\theta}\,\dot{\phi} - r\cos\theta\sin\phi\,\dot{\phi}^2 + r\cos\theta\cos\phi\,\ddot{\phi}\right)\end{array} \tag{75}$$

$$= -2\sin\theta\sin\phi\,\dot{\theta}\,\dot{r} + 2\cos\theta\cos\phi\,\dot{r}\,\dot{\phi} - 2r\sin\theta\cos\phi\,\dot{\theta}\,\dot{\phi} + \cos\theta\sin\phi\,\ddot{r} - r\sin\theta\sin\phi\,\ddot{\theta} + r\cos\theta\cos\phi\,\ddot{\phi} - r\cos\theta\sin\phi\left(\dot{\theta}^2 + \dot{\phi}^2\right) \tag{76}$$

$$\ddot{y} = \begin{array}{l}(\sin\theta\sin\phi\,\ddot{r} + r\cos\theta\sin\phi\,\dot{\theta} + r\cos\phi\,\sin\theta\,\dot{\phi}) + \\ \left(\cos\theta\sin\phi\,\dot{r}\,\dot{\theta} - r\sin\theta\sin\phi\,\dot{\theta}^2 + r\cos\theta\cos\phi\,\dot{\theta}\,\dot{\phi} + r\cos\theta\sin\phi\,\ddot{\theta}\right) + \left(\sin\theta\cos\phi\,\dot{r}\,\dot{\phi} + r\cos\theta\cos\phi\,\dot{\theta}\,\dot{\phi} - r\sin\theta\sin\phi\,\dot{\phi}^2 + r\sin\theta\cos\phi\,\ddot{\phi}\right)\end{array} \tag{77}$$

$$= 2\cos\theta\sin\phi\,\dot{\theta}\,\dot{r} + 2\sin\theta\cos\phi\,\dot{r}\,\dot{\phi} + 2r\cos\theta\cos\phi\,\dot{\theta}\,\dot{\phi} + \sin\theta\sin\phi\,\ddot{r} + r\cos\theta\sin\phi\,\ddot{\theta} + r\sin\theta\cos\phi\,\ddot{\phi} - r\sin\theta\sin\phi\left(\dot{\theta}^2 + \dot{\phi}^2\right) \tag{78}$$

$$\ddot{z} = (\cos\phi\,\ddot{r} - \sin\phi\,\dot{r}\,\dot{\phi}) - \left(\dot{r}\sin\phi\,\dot{\phi} + r\cos\phi\,\dot{\phi}^2 + r\sin\phi\,\ddot{\phi}\right) \tag{79}$$

$$= -r\cos\phi\,\dot{\phi}^2 + \cos\phi\,\ddot{r} - 2\sin\phi\,\dot{\phi}\,\dot{r} - r\sin\phi\,\ddot{\phi}. \tag{80}$$

Plugging these in gives

$$\ddot{\mathbf{r}} = \left(\ddot{r} - r\dot{\phi}^2\right)\begin{bmatrix} \cos\theta\sin\phi \\ \sin\theta\sin\phi \\ \cos\phi \end{bmatrix} + \left(2r\cos\phi\,\dot{\theta}\,\dot{\phi} + 2\sin\phi\,\dot{\theta}\,\dot{r} + r\sin\phi\,\ddot{\theta}\right)\begin{bmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{bmatrix} + \left(2\dot{r}\,\dot{\phi} + r\ddot{\phi}\right)\begin{bmatrix} \cos\theta\cos\phi \\ \sin\theta\cos\phi \\ -\sin\phi \end{bmatrix} - r\sin\phi\,\dot{\theta}^2\begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix}, \tag{81}$$

but

$$\sin\phi\,\hat{\mathbf{r}} + \cos\phi\,\hat{\boldsymbol{\phi}} = \begin{bmatrix} \cos\theta\sin^2\phi + \cos\theta\cos^2\phi \\ \sin\theta\sin^2\phi + \sin\theta\cos^2\phi \\ 0 \end{bmatrix} \tag{82}$$

$$= \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix}, \tag{83}$$

so

$$\ddot{\mathbf{r}} = \left(\ddot{r} - r\dot{\phi}^2\right)\hat{\mathbf{r}} + \left(2r\cos\phi\,\dot{\theta}\,\dot{\phi} + 2\sin\phi\,\dot{\theta}\,\dot{r} + r\sin\phi\,\ddot{\theta}\right)\hat{\boldsymbol{\theta}} + \left(2\dot{r}\,\dot{\phi} + r\ddot{\phi}\right)\hat{\boldsymbol{\phi}} - r\sin\phi\,\dot{\theta}^2\left(\sin\phi\,\hat{\mathbf{r}} + \cos\phi\,\hat{\boldsymbol{\phi}}\right) \tag{84}$$

$$= \left(\ddot{r} - r\dot{\phi}^2 - r\sin^2\phi\,\dot{\theta}^2\right)\hat{\mathbf{r}} + \left(2\sin\phi\,\dot{\theta}\,\dot{r} + 2r\cos\phi\,\dot{\theta}\,\dot{\phi} + r\sin\phi\,\ddot{\theta}\right)\hat{\boldsymbol{\theta}} + \left(2\dot{r}\,\dot{\phi} + r\ddot{\phi} - r\sin\phi\cos\phi\,\dot{\theta}^2\right)\hat{\boldsymbol{\phi}}. \tag{85}$$

Time derivatives of the unit vectors are

$$\dot{\hat{\mathbf{r}}} = \sin\phi\,\dot\theta\,\hat{\boldsymbol\theta} + \dot\phi\,\hat{\boldsymbol\phi} \tag{86}$$

$$\dot{\hat{\boldsymbol\theta}} = -\dot\theta\left(\sin\phi\,\hat{\mathbf{r}} + \cos\phi\,\hat{\boldsymbol\phi}\right) \tag{87}$$

$$\dot{\hat{\boldsymbol\phi}} = -\dot\phi\,\hat{\mathbf{r}} + \cos\phi\,\dot\theta\,\hat{\boldsymbol\theta}. \tag{88}$$

The curl is

$$\nabla\times\mathbf{F} = \frac{1}{r\sin\phi}\left[\frac{\partial}{\partial\phi}(\sin\phi\,F_\theta) - \frac{\partial F_\phi}{\partial\theta}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\phi}\frac{\partial F_r}{\partial\theta} - \frac{\partial}{\partial r}(r\,F_\theta)\right]\hat{\boldsymbol\phi} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r\,F_\phi) - \frac{\partial F_r}{\partial\phi}\right]\hat{\boldsymbol\theta}. \tag{89}$$

The Laplacian is

$$\nabla^2 \equiv \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin^2\phi}\frac{\partial^2}{\partial\theta^2} + \frac{1}{r^2\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial}{\partial\phi}\right) \tag{90}$$

$$= \frac{1}{r^2}\left(r^2\frac{\partial^2}{\partial r^2} + 2r\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin^2\phi}\frac{\partial^2}{\partial\theta^2} + \frac{1}{r^2\sin\phi}\left(\cos\phi\frac{\partial}{\partial\phi} + \sin\phi\frac{\partial^2}{\partial\phi^2}\right) \tag{91}$$

$$= \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2\sin^2\phi}\frac{\partial^2}{\partial\theta^2} + \frac{\cos\phi}{r^2\sin\phi}\frac{\partial}{\partial\phi} + \frac{1}{r^2}\frac{\partial^2}{\partial\phi^2}. \tag{92}$$

The vector Laplacian in spherical coordinates is given by

$$\nabla^2\mathbf{v} = \begin{bmatrix} \frac{1}{r}\frac{\partial^2(r\,v_r)}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial\theta^2} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_r}{\partial\phi^2} + \frac{\cot\theta}{r^2}\frac{\partial v_r}{\partial\theta} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial\theta} - \frac{2}{r^2\sin\theta}\frac{\partial v_\phi}{\partial\phi} - \frac{2\,v_r}{r^2} - \frac{2\cot\theta}{r^2}v_\theta \\ \frac{1}{r}\frac{\partial^2(r\,v_\theta)}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial\theta^2} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_\theta}{\partial\phi^2} + \frac{\cot\theta}{r^2}\frac{\partial v_\theta}{\partial\theta} - \frac{2}{r^2}\frac{\cot\theta}{\sin\theta}\frac{\partial v_\phi}{\partial\phi} + \frac{2}{r^2}\frac{\partial v_r}{\partial\theta} - \frac{v_\theta}{r^2\sin^2\theta} \\ \frac{1}{r}\frac{\partial^2(r\,v_\phi)}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 v_\phi}{\partial\theta^2} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_\phi}{\partial\phi^2} + \frac{\cot\theta}{r^2}\frac{\partial v_\phi}{\partial\theta} + \frac{2}{r^2\sin\theta}\frac{\partial v_r}{\partial\phi} + \frac{2\cot\theta}{r^2\sin\theta}\frac{\partial v_\theta}{\partial\phi} - \frac{v_\phi}{r^2\sin^2\theta} \end{bmatrix}. \tag{93}$$

To express partial derivatives with respect to Cartesian axes in terms of partial derivatives of the spherical coordinates,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} r\cos\theta\sin\phi \\ r\sin\theta\sin\phi \\ r\cos\phi \end{bmatrix} \tag{94}$$

$$\begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} = \begin{bmatrix} \cos\theta\sin\phi\,dr - r\sin\theta\sin\phi\,d\theta + r\cos\theta\cos\phi\,d\phi \\ \sin\theta\sin\phi\,dr + r\sin\phi\cos\theta\,d\theta + r\sin\theta\cos\phi\,d\phi \\ \cos\phi\,dr - r\sin\phi\,d\phi \end{bmatrix} \tag{95}$$

$$= \begin{bmatrix} \cos\theta\sin\phi & -r\sin\theta\sin\phi & r\cos\theta\cos\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\phi & 0 & -r\sin\phi \end{bmatrix}\begin{bmatrix} dr \\ d\theta \\ d\phi \end{bmatrix}. \tag{96}$$

Upon inversion, the result is

$$\begin{bmatrix} dr \\ d\theta \\ d\phi \end{bmatrix} = \begin{bmatrix} \cos\theta\sin\phi & \sin\theta\sin\phi & \cos\phi \\ -\frac{\sin\theta}{r\sin\phi} & \frac{\cos\theta}{r\sin\phi} & 0 \\ \frac{\cos\theta\cos\phi}{r} & \frac{\sin\theta\cos\phi}{r} & -\frac{\sin\phi}{r} \end{bmatrix}\begin{bmatrix} dx \\ dy \\ dz \end{bmatrix}. \tag{97}$$

The Cartesian partial derivatives in spherical coordinates are therefore

$$\frac{\partial}{\partial x} = \cos\theta\sin\phi\frac{\partial}{\partial r} - \frac{\sin\theta}{r\sin\phi}\frac{\partial}{\partial\theta} + \frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial\phi} \tag{98}$$

$$\frac{\partial}{\partial y} = \sin\theta\sin\phi\frac{\partial}{\partial r} + \frac{\cos\theta}{r\sin\phi}\frac{\partial}{\partial\theta} + \frac{\sin\theta\cos\phi}{r}\frac{\partial}{\partial\phi} \tag{99}$$

$$\frac{\partial}{\partial z} = \cos\phi\frac{\partial}{\partial r} - \frac{\sin\phi}{r}\frac{\partial}{\partial\phi} \tag{100}$$

(Gasiorowicz 1974, pp. 167-168; Arfken 1985, p. 108).

The Helmholtz differential equation is separable in spherical coordinates.

SEE ALSO: Azimuth, Colatitude, Great Circle, Helmholtz Differential Equation--Spherical Coordinates, Latitude, Longitude, Oblate Spheroidal Coordinates, Polar Angle, Prolate Spheroidal Coordinates, Zenith Angle

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